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Regularity properties of viscosity solutions of integro-partial differential equations of Hamilton–Jacobi–Bellman type

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Abstract

We study the regularity properties of integro-partial differential equations of Hamilton–Jacobi–Bellman type with the terminal condition, which can be interpreted through a stochastic control system, composed of a forward and a backward stochastic differential equation, both driven by a Brownian motion and a compensated Poisson random measure. More precisely, we prove that, under appropriate assumptions, the viscosity solution of such equations is jointly Lipschitz and jointly semiconcave in $(t, x) \in \Delta \times \mathbb{R}^d$, for all compact time intervals Δ excluding the terminal time. Our approach is based on the time change for the Brownian motion and on Kulik's transformation for the Poisson random measure.

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1. Introduction

We are interested in the regularity properties of the viscosity solution for a certain class of integro-partial differential equations (IPDEs) of Hamilton–Jacobi–Bellman (HJB) type. In order

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to be more precise, let us consider the following possibly degenerate equation:

$$\begin{cases} \dfrac{\partial}{\partial t}V(t,x) + \inf\limits_{u\in U}\{(\mathcal{L}^u + B^u)V(t,x) + f(t,x,V(t,x),(D_xV\sigma)(t,x), \\ \quad V(t,x+\beta(t,x,u,\cdot)) - V(t,x),u)\} = 0; \\ V(T,x) = \Phi(x), \end{cases} \tag{1.1}$$

where U is a compact metric space, $\mathcal{L}^u$ is the linear second order differential operator

$$\mathcal{L}^u\varphi(x) = \mathrm{tr}\left(\frac{1}{2}\sigma\sigma^T(t,x,u)D^2_{xx}\varphi(x)\right) + b(t,x,u)\cdot D_x\varphi(x), \quad \varphi \in C^2(\mathbb{R}^d),$$

and B^u is the integro-differential operator:

$$B^u\varphi(x) = \int_E [\varphi(x+\beta(t,x,u,e)) - \varphi(x) - \beta(t,x,u,e)\cdot D_x\varphi(x)]\Pi(\mathrm{d}e),$$
$$\varphi \in C^1_b(\mathbb{R}^d).$$

Here, Π denotes a finite Lévy measure on $E = \mathbb{R}^n \setminus \{0\}$. Our main results say that, under appropriate assumptions, for all $\delta > 0$, the viscosity solution V is jointly Lipschitz and jointly semiconcave on $[0, T-\delta] \times \mathbb{R}^d$, i.e., there is some constant C_δ such that

$$|V(t_0,x_0) - V(t_1,x_1)| \le C_\delta(|t_0-t_1| + |x_0-x_1|),$$
$$\begin{aligned}\lambda V(t_0,x_0) + (1-\lambda)V(t_1,x_1) \ \le\ & V(\lambda(t_0,x_0) + (1-\lambda)(t_1,x_1)) \\ & + C_\delta\lambda(1-\lambda)(|t_0-t_1|^2 + |x_0-x_1|^2),\end{aligned}$$

for all $(t_0,x_0),(t_1,x_1) \in [0,T-\delta]\times\mathbb{R}^d$. The joint semiconcavity of V stems its importance from the fact that, due to Alexandrov's theorem, it implies that V has a second order expansion in (t,x), $\mathrm{d}t\mathrm{d}x$-a.e. Such expansions are important, for instance, for the study of the propagation of singularities.

Although, at least for PDEs of HJB type, the regularity of the solution of strictly elliptic equations (with $\sigma\sigma^T \ge \alpha I$, for $\alpha > 0$) has been well understood for a long time, the joint regularity (Lipschitz continuity and semiconcavity) in (t,x) for the viscosity solution of such equations, for which $\sigma\sigma^T$ is not necessarily strictly elliptic, have been studied only recently. However, under suitable hypotheses, the Lipschitz continuity and the semiconcavity of $V(t,x)$ in x as well as the Hölder continuity of $V(t,x)$ in t has already been known for a longer time. As concerns the semiconcavity of $V(t,x)$ in x, a purely analytical proof was given by Ishii and Lions [6]; for a stochastic proof the reader is referred, for example, to Yong and Zhou [14]. Concerning the Lipschitz continuity of V in x and the Hölder continuity in t (with Hölder coefficient 1/2), we refer, for example, to Pham [10]. Krylov [7] suggested the joint Lipschitz continuity of $V(t,x)$ on (t,x). However, counterexamples show that, in general, one cannot get the Lipschitz continuity or semiconcavity in (t,x) for the whole domain $[0,T]\times\mathbb{R}^d$. In [2], it was shown that the viscosity solution of PDEs of HJB type (with $\beta = 0$) is Lipschitz and semiconcave over $[0,T-\delta]\times\mathbb{R}^d$ for $\delta > 0$. In [3], these results were extended to PDEs with obstacle. The approach in [2,3] consists in the study of the viscosity solution V with the help of its stochastic interpretation as a value function of an associated stochastic control problem; it uses, in particular, the method of time change, which was translated to backward stochastic differential equations (BSDEs) in [3].

In this paper we study the joint regularity of $V(t,x)$ in (t,x) through the stochastic interpretation of the above HJB equation as a stochastic control problem composed of a forward and a backward stochastic differential equation (SDE). More precisely, let $(t,x) \in [0,T] \times \mathbb{R}^d$, $B = (B_s)_{s\in[t,T]}$ be a d-dimensional Brownian motion with initial value zero at time t, and let μ be a Poisson random measure on $[t,T] \times E$. We denote by $\mathbb{F}$ the filtration generated by B and μ, and by $\mathcal{U}^{B,\mu}(t,T)$ the set of all $\mathbb{F}$-predictable control processes with values in U. It is by now standard that the SDE driven by the Brownian motion B and the compensated Poisson random measure $\tilde{\mu}$:

$$X_s^{t,x,u} = x + \int_t^s b(r, X_r^{t,x,u}, u_r)\mathrm{d}r + \int_t^s \sigma(r, X_r^{t,x,u}, u_r)\mathrm{d}B_r \\ + \int_t^s \int_E \beta(r, X_{r-}^{t,x,u}, u_r, e)\tilde{\mu}(\mathrm{d}r, \mathrm{d}e), \quad s \in [t,T], \tag{1.2}$$

has a unique solution under appropriate assumptions for the coefficients. With this SDE we associate the BSDE with jumps

$$Y_s^{t,x,u} = \Phi(X_T^{t,x,u}) + \int_s^T f(r, X_r^{t,x,u}, Y_r^{t,x,u}, Z_r^{t,x,u}, U_r^{t,x,u}, u_r)\mathrm{d}r - \int_s^T Z_r^{t,x,u}\mathrm{d}B_r \\ - \int_s^T \int_E U_r^{t,x,u}(e)\tilde{\mu}(\mathrm{d}r, \mathrm{d}e), \quad s \in [t,T]. \tag{1.3}$$

(As concerns the assumptions on the coefficients, we refer to the hypotheses (**H1**)–(**H5**) in Sections 2 and 3.) Such kind of BSDEs with jumps were first studied by Tang and Li [12] in 1994. From Tang and Li [12] or Barles et al. [1] we know that the above BSDE with jumps (1.3) has a unique square integrable solution $(Y^{t,x,u}, Z^{t,x,u}, U^{t,x,u})$. Moreover, since $Y^{t,x,u}$ is $\mathbb{F}$-adapted, $Y_t^{t,x,u}$ is deterministic. It follows from Barles et al. [1] or Pham [10] that the value function

$$V(t,x) = \inf_{u\in\mathcal{U}^{B,\mu}(t,T)} Y_t^{t,x,u}, \quad (t,x) \in [0,T] \times \mathbb{R}^d \tag{1.4}$$

is the viscosity solution of our IPDE.

Since unlike [2,3], our system involves not only the Brownian motion B but also the Poisson random measure μ, the method of time change for the Brownian motion alone is not sufficient for our approach here. So we combine the method of time change for the Brownian motion by Kulik's transformation for Poisson random measures (see, [8,9]). To our best knowledge, the use of Kulik's transformation for the study of stochastic control problems is new. Because of the difficulty to obtain suitable L^p-estimates of the stochastic integrals with respect the compensated Poisson random measure (see, for example, [10]) we have to restrict ourselves to the case of a finite Lévy measure $\Pi(E) < +\infty$. The more general case where $\int_E (1 \wedge |e|^2)\Pi(\mathrm{d}e) < +\infty$ remains still open.

Our paper is organized as follows. In Section 2 we introduce our main tools, i.e., the method of time change for the Brownian motion and Kulik's transformation for the Poisson random measure, with the help of which we study the joint Lipschitz continuity for the viscosity solution of the IPDEs of HJB type. This method of time change for the Brownian motion combined with Kulik's transformation is extended in Section 3 to the study of the semiconcavity property for the viscosity solution of IPDE (1.1). The proof of more technical statements and estimates used in Section 3 is shifted in the Appendix.

2. Lipschitz continuity

In this section, we prove the joint Lipschitz continuity of the viscosity solution of a certain class of integro-differential Hamilton–Jacobi–Bellman (HJB) equations.

Let T be an arbitrarily fixed time horizon, U a compact metric space, $E = \mathbb{R}^d \setminus \{0\}$ and $\mathcal{B}(E)$ be the Borel σ-algebra over E. We are concerned with the integro-partial differential equation of HJB type (1.1). The coefficients

$$\begin{aligned}&b : [0, T] \times \mathbb{R}^d \times U \to \mathbb{R}^d, \qquad \sigma : [0, T] \times \mathbb{R}^d \times U \to \mathbb{R}^{d\times d},\\ &\beta : [0, T] \times \mathbb{R}^d \times U \times E \to \mathbb{R}^d,\\ &f : [0, T] \times \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d \times L^2(E, \mathcal{B}(E), \Pi; \mathbb{R}) \times U \to \mathbb{R} \quad \text{and} \quad \Phi : \mathbb{R}^d \to \mathbb{R}\end{aligned}$$

are bounded continuous functions which satisfy the following conditions.

(**H1**) There exists a constant $K > 0$ such that, for any $\xi_i = (s_i, y_i) \in [0, T] \times \mathbb{R}^d, u \in U$, $i = 1, 2$,

$$\begin{aligned}&|b(\xi_1, u) - b(\xi_2, u)| + |\sigma(\xi_1, u) - \sigma(\xi_2, u)| + \left(\int_E |\beta(\xi_1, u, e) - \beta(\xi_2, u, e)|^4 \Pi(\mathrm{d}e)\right)^{1/4}\\ &\quad \le K(|s_1 - s_2| + |y_1 - y_2|).\end{aligned}$$

(**H2**) The function f is Lipschitz in (t, x, y, z, p), uniformly with respect to $u \in U$, and the function Φ is a Lipschitz function.

The integro-PDE (1.1), as is well-known by now (see, for instance, [1]), has a unique continuous viscosity solution $V(t, x)$ in the class of the continuous functions with at most polynomial growth.

Let $\{B^0_s\}_{s\ge0}$ be a d-dimensional Brownian motion defined on a complete space $(\Omega_1, \mathcal{F}_1, \mathbb{P}_1)$, and η be a Poisson random measure defined on a complete probability space $(\Omega_2, \mathcal{F}_2, \mathbb{P}_2)$. We introduce $(\Omega, \mathcal{F}, \mathbb{P})$ as the product space $(\Omega, \mathcal{F}, \mathbb{P}) = (\Omega_1, \mathcal{F}_1, \mathbb{P}_1) \otimes (\Omega_2, \mathcal{F}_2, \mathbb{P}_2) = (\Omega_1 \times \Omega_2, \mathcal{F}_1 \otimes \mathcal{F}_2, \mathbb{P}_1 \otimes \mathbb{P}_2)$. The processes B^0 and η are canonically extended from $(\Omega_1, \mathcal{F}_1, \mathbb{P}_1)$ and $(\Omega_2, \mathcal{F}_2, \mathbb{P}_2)$, respectively, to the product space $(\Omega, \mathcal{F}, \mathbb{P})$. We denote the compensated Poisson random measure associated with η by $\tilde\eta$, i.e., $\tilde\eta(\mathrm{d}t, \mathrm{d}e) = \eta(\mathrm{d}t, \mathrm{d}e) - \mathrm{d}t\,\Pi(\mathrm{d}e)$. We assume throughout this paper that the Lévy measure Π is a finite measure on $(E, \mathcal{B}(E))$.

We define the process $\{B_s\}_{s\ge t}$ by putting

$$B_s = B^0_s - B^0_t, \quad s \in [t, T], \tag{2.1}$$

so that $\{B_s\}_{s\ge t}$ is a Brownian motion beginning at time t with $B_t = 0$. Furthermore, we denote by μ the restriction of the Poisson random measure η from $[0, T] \times E$ to $[t, T] \times E$, and by $\tilde\mu$ its compensated measure.

We put

$$\begin{aligned}\mathcal{F}^B_s &= \sigma\{B_r, r \in [t, s]\} \vee \mathcal{N}_{\mathbb{P}_1},\\ \mathcal{F}^\mu_s &= \sigma\{\mu((t, r] \times \Delta) : \Delta \in \mathcal{B}(E), r \in [t, s]\} \vee \mathcal{N}_{\mathbb{P}_2},\end{aligned}$$

and

$$\mathcal{F}_s = (F^B_s \otimes \mathcal{F}^\mu_s) \vee \mathcal{N}_{\mathbb{P}}, \quad s \in [t, T],$$

where $\mathcal{N}_{\mathbb{P}_1}, \mathcal{N}_{\mathbb{P}_2}$ and $\mathcal{N}_{\mathbb{P}}$ are the collections of the null sets under the corresponding probability measure.

Let us also introduce the following spaces of stochastic processes over $(\Omega, \mathcal{F}, \mathbb{P})$ which will be needed in what follows. By $\mathcal{S}^2(t, T; \mathbb{R}^d)$ we denote the set of all $\mathbb{F}$-adapted càdlàg processes $\{Y_s; t \le s \le T\}$ such that

$$\|Y\|_{\mathcal{S}^2(t,T;\mathbb{R}^d)} = \mathbb{E}\left[\sup_{t\le s\le T} |Y_s|^2\right] < \infty.$$

Let $\mathbb{L}^2(t, T; \mathbb{R}^d)$ denote the set of all $\mathbb{F}$-predictable d-dimensional processes $\{Z_s : t \le s \le T\}$ such that

$$\|Z\|_{\mathbb{L}^2(t,T;\mathbb{R}^d)} = \left(\mathbb{E}\left[\int_t^T |Z_s|^2 \mathrm{d}s\right]\right)^{1/2} < \infty.$$

Finally, we also introduce the space $\mathbb{L}^2(t, T; \tilde{\mu}, \mathbb{R})$ of mappings $U : \Omega \times [0, T] \times E \to \mathbb{R}$ which are $\mathbb{F}$-predictable and measurable such that

$$\|U\|_{\mathbb{L}^2(t,T;\tilde{\mu},\mathbb{R})} = \left(\mathbb{E}\left[\int_t^T \int_E |U_s(e)|^2 \Pi(\mathrm{d}e)\mathrm{d}s\right]\right)^{1/2} < \infty.$$

Let us now consider the following stochastic differential equation driven by the Brownian motion B and the compensated Poisson random measure $\tilde{\mu}$:

$$\begin{aligned} X_s^{t,x,u} &= x + \int_t^s b(r, X_r^{t,x,u}, u_r)\mathrm{d}r + \int_t^s \sigma(r, X_r^{t,x,u}, u_r)\mathrm{d}B_r \\ &\quad + \int_t^s \int_E \beta(r, X_{r-}^{t,x,u}, u_r, e)\tilde{\mu}(\mathrm{d}r, \mathrm{d}e), \qquad s \in [t, T], \end{aligned} \tag{2.2}$$

where the process $u : [t, T] \times \Omega \to U$ is an admissible control, i.e., an $\mathbb{F}$-predictable process with values in U; the space of admissible controls over the time interval $[t, T]$ is denoted by $\mathcal{U}^{B,\mu}(t, T)$. The following theorem is by now classical.

Theorem 2.1. *Assume the Lipschitz condition* (**H1**). *For any fixed admissible control $u(\cdot) \in \mathcal{U}(t, T)$, there exists a unique adapted càdlàg solution $(X_s^{t,x,u})_{s\in[t,T]} \in \mathcal{S}^2(t, T; \mathbb{R}^d)$ of the stochastic differential equation* (2.2).

We associate SDE (2.2) with the backward stochastic differential equation

$$\begin{aligned} Y_s^{t,x,u} &= \Phi(X_T^{t,x,u}) + \int_s^T f(r, X_r^{t,x,u}, Y_r^{t,x,u}, Z_r^{t,x,u}, U_r^{t,x,u}, u_r)\mathrm{d}r - \int_s^T Z_r^{t,x,u}\mathrm{d}B_r \\ &\quad - \int_s^T \int_E U_r^{t,x,u}(e)\tilde{\mu}(\mathrm{d}r, \mathrm{d}e), \quad s \in [t, T]. \end{aligned}$$

Then from Barles et al. [1], Tang and Li [12], we know that this BSDE has a unique solution

$$(Y^{t,x,u}, Z^{t,x,u}, U^{t,x,u}) \in \mathcal{S}^2(t, T; \mathbb{R}) \times \mathbb{L}^2(t, T; \mathbb{R}^d) \times \mathbb{L}^2(t, T; \tilde{\mu}, \mathbb{R}).$$

Notice that $Y_t^{t,x,u}$ is $\mathcal{F}_t$-measurable, hence it is deterministic in the sense that it coincides $\mathbb{P}$-a.s. with a real constant, with which it is identified. Thus, we have

$$Y_t^{t,x,u} = \mathbb{E}[Y_t^{t,x,u}] = \mathbb{E}\left[\int_t^T f(r, X_r^{t,x,u}, Y_r^{t,x,u}, Z_r^{t,x,u}, U_r^{t,x,u}, u_r)\mathrm{d}r + \Phi(X_T^{t,x,u})\right].$$

As usual in stochastic control problems, we define the cost functional $J(t,x;u)$ associated with $u \in \mathcal{U}^{B,\mu}(0,T)$ by setting $J(t,x;u) := Y_t^{t,x,u}$, and the value function is defined as follows:

$$V(t,x) = \inf_{u(\cdot)\in\mathcal{U}^{B,\mu}(t,T)} J(t,x;u), \quad (t,x) \in [0,T] \times \mathbb{R}^d.$$

It is well known by now that $V = \{V(t,x) : (t,x) \in [0,T] \times \mathbb{R}^d\}$ is a continuous viscosity solution of the HJB equation (1.1). Moreover, V is the unique viscosity solution in the class of continuous functions with at most polynomial growth (see: [10,13]).

Our main result in this section is the following theorem.

Theorem 2.2. *Let $\delta \in (0,T)$ be arbitrary but fixed. Under our assumptions* (**H1**) *and* (**H2**)*, the value function $V(\cdot,\cdot)$ is jointly Lipschitz continuous on $[0,T-\delta] \times \mathbb{R}^d$, i.e., for some constant C_δ we have, for all $(t_0,x_0),(t_1,x_1) \in [0,T-\delta] \times \mathbb{R}^d$:*

$$|V(t_0,x_0) - V(t_1,x_1)| \le C_\delta(|t_0 - t_1| + |x_0 - x_1|).$$

Remark 2.3. In general we cannot expect to get the joint Lipschitz continuity over the whole domain $[0,T] \times \mathbb{R}^d$. In [2] is given an easy counterexample: we study the problem

$$X_s^{t,x} = x + B_s, \quad s \in [t,T],\ x \in \mathbb{R};$$
$$Y_s^{t,x} = -\mathbb{E}[|X_T^{t,x}||\mathcal{F}_s] = -\mathbb{E}[|x + B_T||\mathcal{F}_s], \quad s \in [t,T],$$

without control neither jumps. Then

$$V(t,x) = Y_t^{t,x} = -\mathbb{E}[|x + B_T|],$$

and, for $x = 0$, recalling that B is a Brownian motion with $B_t = 0$, we have

$$V(t,0) = -\mathbb{E}[|B_T|] = -\sqrt{\frac{2}{\pi}}\sqrt{T-t}, \quad t \in [0,T].$$

Obviously, $V(\cdot,x)$ is not Lipschitz in t for $t = T$. However, V is jointly Lipschitz on $[0,T-\delta] \times \mathbb{R}$, for $\delta \in (0,T)$.

Let us introduce now Kulik's transformation in our framework. The reader interested in more details on this transformation is referred to the papers [8,9].

Let $t_0, t_1 \in [0,T]$ and let, for $t = t_0$, μ be the Poisson random measure which we have introduced as restriction of η from $[0,T] \times E$ to $[t_0,T] \times E$. With the help of μ we define now a random measure $\tau(\mu)$ on $[t_0,T] \times E$. Denoting by

$$\tau : [t_1,T] \to [t_0,T]$$

the linear time change

$$\tau(s) = t_0 + \frac{T-t_0}{T-t_1}(s - t_1), \quad s \in [t_1,T],$$

we put

$$\tau(\mu)([t_1,s] \times \Delta) := \mu([\tau(t_1),\tau(s)] \times \Delta), \quad t_1 \le s \le T,\ \Delta \in \mathcal{B}(E).$$

Observing that $\dot\tau = \dot\tau(s) = \frac{T-t_0}{T-t_1}$, we put

$$\gamma = \ln(\dot\tau) = \ln\left(\frac{T-t_0}{T-t_1}\right).$$

From Lemma 1.1 in [8] we know that, for all $\{s_1, \dots, s_n\} \subset [t_1, T]$, $\Delta_1, \dots, \Delta_n \in \mathcal{B}(E)$ and all Borel function $\varphi : \mathbb{R}^n \to \mathbb{R}^+$, we have the following.

Lemma 2.4.

$$\begin{aligned}&\mathbb{E}[\varphi(\tau(\mu)([t_1, s_1] \times \Delta_1), \dots, \tau(\mu)([t_1, s_n] \times \Delta_n))]\\ &\quad = \mathbb{E}[\rho_\tau \varphi(\eta([t_1, s_1] \times \Delta_1), \dots, \eta([t_1, s_n] \times \Delta_n))],\end{aligned}$$

where

$$\rho_\tau = \exp\{\gamma \eta([t_1, T] \times E) - (t_1 - t_0)\Pi(E)\}.$$

For the convenience of the reader we sketch the proof. However, we restrict to a special case ($n = 1$), the proof of the general case $n \geq 1$ can be carried out with a similar argument and can be consulted for the more general case $\Pi(E) = +\infty$ in [8].

Proof (*For* $n = 1$)**.** Observing that for $\Delta_2 = E \setminus \Delta_1$,

$$\eta([t_1, s_1] \times \Delta_1), \qquad \eta([s_1, T] \times \Delta_1) \quad \text{and} \quad \eta([t_1, T] \times \Delta_2)$$

are independent Poisson distributed random variables with the intensities $(s_1 - t_1)\Pi(\Delta_1)$, $(T - s_1)\Pi(\Delta_1)$ and $(T - t_1)\Pi(\Delta_2)$, respectively, we have

$$\begin{aligned}\mathbb{E}[\rho_\tau \varphi(\eta([t_1, s_1] \times \Delta_1))] &= \left\{\sum_{k,l \geq 0} \varphi(k) \exp\{\gamma k + \gamma l - (t_1 - t_0)\Pi(\Delta_1)\}\right.\\ &\quad \left. \times \exp\{-(T - t_1)\Pi(\Delta_1)\} \frac{((s_1 - t_1)\Pi(\Delta_1))^k}{k!} \frac{((T - s_1)\Pi(\Delta_1))^l}{l!}\right\}\\ &\quad \times \left\{\sum_{m \geq 0} \exp\{\gamma m - (t_1 - t_0)\Pi(\Delta_2)\} \exp\{-(T - t_1)\Pi(\Delta_2)\} \frac{((T - t_1)\Pi(\Delta_2))^m}{m!}\right\}\\ &= I_1 \times I_2.\end{aligned}$$

But, taking into account the definition of γ and that of $\tau(s_1)$ we have

$$\begin{aligned}I_1 &= \left\{\sum_{k \geq 0} \varphi(k) \exp\{-(T - t_0)\Pi(\Delta_1)\} \frac{1}{k!} \left(\left(\frac{T - t_0}{T - t_1}\right)(s_1 - t_1)\Pi(\Delta_1)\right)^k\right\}\\ &\quad \times \left\{\sum_{l \geq 0} \frac{1}{l!} \left(\left(\frac{T - t_0}{T - t_1}\right)(T - s_1)\Pi(\Delta_1)\right)^l\right\}\\ &= \left\{\sum_{k \geq 0} \varphi(k) \exp\{-(T - t_0)\Pi(\Delta_1)\} \frac{1}{k!} ((\tau(s_1) - t_0)\Pi(\Delta_1))^k\right\}\\ &\quad \times \exp\{(T - \tau(s_1))\Pi(\Delta_1)\}\\ &= \sum_{k \geq 0} \varphi(k) \exp\{-(\tau(s_1) - t_0)\Pi(\Delta_1)\} \frac{1}{k!} ((\tau(s_1) - t_0)\Pi(\Delta_1))^k\\ &= \mathbb{E}[\varphi(\tau(\mu)([t_1, s_1] \times \Delta_1))].\end{aligned}$$

In analogy to the computation for I_1, but now with $\varphi \equiv 1$, we get that $I_2 = 1$. Consequently,

$$\mathbb{E}[\rho_\tau \varphi(\eta([t_1, s_1] \times \Delta_1))] = \mathbb{E}[\varphi(\tau(\mu)([t_1, s_1] \times \Delta_1))].$$

Hence the proof is complete. $\square$

From the above lemma we have, for all $n \geq 1$, $\{s_1, \ldots, s_n\} \subset [t_1, T]$, $\Delta_1, \ldots, \Delta_n \in \mathcal{B}(E)$ and $\varphi : \mathbb{R}^n \to \mathbb{R}^+$ Borel function,

$$\begin{aligned}
&\mathbb{E}[\varphi(\eta([t_1, s_1] \times \Delta_1), \ldots, \eta([t_1, s_n] \times \Delta_n))] \\
&\quad = \mathbb{E}\big[\rho_\tau \varphi(\eta([t_1, s_1] \times \Delta_1), \ldots, \eta([t_1, s_n] \times \Delta_n)) \\
&\qquad \times \exp\{-\gamma \eta([t_1, T] \times E) + (t_1 - t_0)\Pi(E)\}\big] \\
&\quad = \mathbb{E}\big[\varphi(\tau(\mu)([t_1, s_1] \times \Delta_1), \ldots, \eta([t_1, s_n] \times \Delta_n)) \\
&\qquad \times \exp\{-\gamma \tau(\mu)([t_1, T] \times E) + (t_1 - t_0)\Pi(E)\}\big] \\
&\quad = \mathbb{E}[g_\tau \varphi(\tau(\mu)([t_1, s_1] \times \Delta_1), \ldots, \eta([t_1, s_n] \times \Delta_n))],
\end{aligned}$$

for $g_\tau = \exp\{-\gamma \tau(\mu)([t_1, T] \times E) + (t_1 - t_0)\Pi(E)\}$.

This allows to show that under the probability measure $\mathbb{Q}_\tau = g_\tau \mathbb{P}$, the point process $\tau(\mu)$ defined over $[t_1, T] \times E$, has the same law as the Poisson random measure η restricted to $[t_1, T] \times E$, under $\mathbb{P}$. Consequently, under $\mathbb{Q}_\tau = g_\tau \mathbb{P}$, $\tau(\eta)$ is a Poisson random measure with compensator $\mathrm{d}s\,\Pi(\mathrm{d}e)$.

We use the same time change $\tau : [t_1, T] \to [t_0, T]$ in order to introduce the process

$$W_t = \frac{1}{\sqrt{\dot{\tau}}} B_{\tau(t)}, \quad t \in [t_1, T].$$

We observe that $W = (W_t)_{t \in [t_1, T]}$ is a Brownian motion under the probability $\mathbb{P}$ but also under $\mathbb{Q}_\tau = g_\tau \mathbb{P}$ (indeed, B and g_τ are independent under $\mathbb{P}$), and W and $\tau(\mu)$ are independent under both $\mathbb{P}$ and $\mathbb{Q}_\tau$.

Let $\varepsilon > 0$. From the definition of the value function V,

$$V(t_0, x_0) = \inf_{u \in \mathcal{U}^{B,\mu}(t_0, T)} J(t_0, x_0, u),$$

we get the existence of an admissible control $u^0 \in \mathcal{U}^{B,\mu}(t_0, T)$ such that

$$J(t_0, x_0, u^0) \leq V(t_0, x_0) + \varepsilon.$$

We define

$$u^1(t) = u^0(\tau(t)), \quad t \in [t_1, T].$$

Then, obviously, $u^1 \in \mathcal{U}^{W, \tau(\mu)}(t_1, T)$, i.e., u^1 is a U-valued process predictable with respect to the filtration

$$\mathcal{F}_t^{W, \tau(\mu)} = \sigma\{W_s, \tau(\mu)([t_1, s] \times \Delta), s \in [t_1, t], \Delta \in \mathcal{B}(E)\} \vee \mathcal{N}_{\mathbb{P}}, \quad t \in [t_1, T],$$

generated by W and $\tau(\mu)$.

Let now $X^0 = \{X_s^0\}_{s \in [t_0, T]}$ be the solution of the forward equation

$$\begin{aligned}
X_s^0 &= x_0 + \int_{t_0}^s b(r, X_r^0, u_r^0)\mathrm{d}r + \int_{t_0}^s \sigma(r, X_r^0, u_r^0)\mathrm{d}B_r \\
&\quad + \int_{t_0}^s \int_E \beta(r, X_{r-}^0, u_r^0, e)\tilde{\mu}(\mathrm{d}r, \mathrm{d}e), \quad s \in [t_0, T],
\end{aligned} \tag{2.3}$$

under the probability $\mathbb{P}$, and let $X^1 = \{X_s^1\}_{s\in[t_1,T]}$ be the solution of the equation

$$X_s^1 = x_1 + \int_{t_1}^s b(r, X_r^1, u_r^1)\mathrm{d}r + \int_{t_1}^s \sigma(r, X_r^1, u_r^1)\mathrm{d}W_r + \int_{t_1}^s \int_E \beta(r, X_{r-}^1, u_r^1, e)\widetilde{\tau(\mu)}^{\mathbb{Q}_\tau}(\mathrm{d}r, \mathrm{d}e), \quad s \in [t_1, T], \tag{2.4}$$

under probability measure $\mathbb{Q}_\tau$. Notice that the compensated Poisson random measure $\tilde{\mu}$ under $\mathbb{P}$ is of the form

$$\tilde{\mu}(\mathrm{d}s, \mathrm{d}e) = \mu(\mathrm{d}s, \mathrm{d}e) - \mathrm{d}s\Pi(\mathrm{d}e), \quad (s, e) \in [t_0, T] \times E,$$

while the compensated Poisson random measure for $\tau(\mu)$ under $\mathbb{Q}_\tau$ has the form

$$\widetilde{\tau(\mu)}^{\mathbb{Q}_\tau}(\mathrm{d}s, \mathrm{d}e) = \tau(\mu)(\mathrm{d}s, \mathrm{d}e) - \mathrm{d}s\Pi(\mathrm{d}e), \quad (s, e) \in [t_1, T] \times E.$$

We employ the BSDE method to prove the Lipschitz continuity of the value function V. For this we associate the above SDEs with the following BSDEs with jumps:

$$Y_s^0 = \Phi(X_T^0) + \int_s^T f(r, X_r^0, Y_r^0, Z_r^0, U_r^0, u_r^0)\mathrm{d}r - \int_s^T Z_r^0 \mathrm{d}B_r - \int_s^T \int_E U_r^0(e)\tilde{\mu}(\mathrm{d}r, \mathrm{d}e), \quad s \in [t_0, T], \tag{2.5}$$

under probability $\mathbb{P}$, and

$$Y_s^1 = \Phi(X_T^1) + \int_s^T f(r, X_r^1, Y_r^1, Z_r^1, U_r^1, u_r^1)\mathrm{d}r - \int_s^T Z_r^1 \mathrm{d}W_r - \int_s^T \int_E U_r^1(e)\widetilde{\tau(\mu)}^{\mathbb{Q}_\tau}(\mathrm{d}r, \mathrm{d}e), \quad s \in [t_1, T], \tag{2.6}$$

under probability $\mathbb{Q}_\tau$. From [1], we know the above two BSDEs have unique solutions $(Y^0, Z^0, U^0) = (Y_s^0, Z_s^0, U_s^0)_{s\in[t_0,T]}$ and $(Y^1, Z^1, U^1) = (Y_s^1, Z_s^1, U_s^1)_{s\in[t_1,T]}$, respectively. While Y^0 is adapted and Z^0 and U^0 are predictable with respect to the filtration generated by B and μ, Y^1 is adapted and Z^1 and U^1 are predictable with respect the filtration generated by W and $\tau(\mu)$. Thus, $Y_{t_0}^0$ and $Y_{t_1}^1$ are deterministic, and from the definition of the cost functionals we have

$$Y_{t_0}^0 - \varepsilon = J(t_0, x_0; u^0) - \varepsilon \le V(t_0, x_0) \left(= \inf_{u\in\mathcal{U}^{B,\mu}(t_0,T)} J(t_0, x_0; u) \right), \tag{2.7}$$

and

$$Y_{t_1}^1 = J(t_1, x_1; u^1) \ge V(t_1, x_1) \left(= \inf_{u\in\mathcal{U}^{W,\tau(\mu)}(t_1,T)} J(t_1, x_1; u) \right). \tag{2.8}$$

Here we have used that the stochastic interpretation of V does not depend on the special choice of the underlying driving Brownian motion and the underlying Poisson random measure with compensator $\mathrm{d}s\Pi(\mathrm{d}e)$. In order to show the Lipschitz property of V in (t, x), we have to estimate

$$\begin{aligned} V(t_0, x_0) - V(t_1, x_1) &\ge J(t_0, x_0; u^0) - J(t_1, x_1; u^1) - \varepsilon \\ &= Y_{t_0}^0 - Y_{t_1}^1 - \varepsilon. \end{aligned}$$

However, in order to estimate the difference between the processes Y^0 and Y^1, we have to make their both BSDEs comparable, i.e., we need them over the same time interval, driven by the same Brownian motion and by the same compensated Poisson random measure. For this reason we apply to SDE (2.4) and BSDE (2.6) the inverse time change $\tau^{-1}:[t_0,T]\to[t_1,T]$. So we introduce the process $\widetilde{X}^1=\{\widetilde{X}^1_s\}_{s\in[t_0,T]}$ by setting $\widetilde{X}^1_s=X^1_{\tau^{-1}(s)}$. We also observe that $W_{\tau^{-1}(r)}=\frac{1}{\sqrt{\dot\tau}}B_r$ and $u^1_{\tau^{-1}(r)}=u^0_r$, $r\in[t_0,T]$. Obviously, $\widetilde{X}^1\in\mathcal{S}^2(t_0,T;\mathbb{R})$ is the unique solution of the SDE

$$
\begin{aligned}
\widetilde{X}^1_s &= x_1+\int_{t_0}^{s} b(\tau^{-1}(r),\widetilde{X}^1_r,u^1_{\tau^{-1}(r)})\mathrm{d}\tau^{-1}(r)+\int_{t_0}^{s}\sigma(\tau^{-1}(r),\widetilde{X}^1_r,u^1_{\tau^{-1}(r)})\mathrm{d}W_{\tau^{-1}(r)}\\
&\quad+\int_{t_0}^{s}\int_E \beta(\tau^{-1}(r),\widetilde{X}^1_{r-},u^1_{\tau^{-1}(r)},e)\widetilde{\tau(\mu)}^{\mathbb{Q}_\tau}(\mathrm{d}\tau^{-1}(r),\mathrm{d}e)\\
&= x_1+\int_{t_0}^{s}\frac{1}{\dot\tau}b(\tau^{-1}(r),\widetilde{X}^1_r,u^0_r)\mathrm{d}r+\int_{t_0}^{s}\frac{1}{\sqrt{\dot\tau}}\sigma(\tau^{-1}(r),\widetilde{X}^1_r,u^0_r)\mathrm{d}B_r\\
&\quad+\int_{t_0}^{s}\int_E \beta(\tau^{-1}(r),\widetilde{X}^1_{r-},u^0_r,e)\left(\tilde\mu(\mathrm{d}r,\mathrm{d}e)+\left(1-\frac{1}{\dot\tau}\right)\Pi(\mathrm{d}e)\mathrm{d}r\right).
\end{aligned}
\tag{2.9}
$$

This time change in Eq. (2.4) makes the processes X^0 and $\widetilde{X}^1$ comparable. More precisely, we have the following.

Lemma 2.5. *There exists some constant C_δ, only depending on the bounds of σ,b,β, their Lipschitz constants, as well as on $\Pi(E)$ and δ, such that, for all $t\in[t_0,T]$,*

$$
\mathbb{E}\left[\sup_{t\le s\le T}|X^0_s-\widetilde{X}^1_s|^2|\mathcal{F}_t\right]\le C_\delta(|t_0-t_1|^2+|X^0_t-\widetilde{X}^1_t|^2).
$$

For the proof of this lemma we need the following estimates obtained by an elementary straightforward computation (see [2,3]).

Lemma 2.6. *There is a constant C_δ only depending on $\delta>0$, such that for all $r\in[t_0,T]$, we have*

$$
\left|1-\frac{1}{\dot\tau}\right|+|\tau^{-1}(r)-r|+|1-\sqrt{\dot\tau}|\le C_\delta|t_0-t_1|.
$$

Remark 2.7. We notice that in the above lemma, the constant C_δ depends heavily on δ. In fact, one can verify that when δ approaches zero, C_δ tends to infinity. That is, the constant C_δ is finite only when $\delta\neq 0$. It is also the reason that we need to emphasize that, in our main results, δ is always assumed to be greater than zero.

Proof of Lemma 2.5. By taking the difference between the SDEs (2.3) and (2.9) and after the conditional expectation of the supremum of its square, we get from Lemma 2.6 and the assumptions on the coefficients, for $s\in[t,T]$,

$$
\begin{aligned}
&\mathbb{E}\left[\sup_{r\in[t,s]}\left|X^0_r-\widetilde{X}^1_r\right|^2\middle|\mathcal{F}_t\right]\\
&=\mathbb{E}\Bigg[\Bigg(\left|X^0_t-\widetilde{X}^1_t\right|+\int_t^s\left|b(v,X^0_v,u^0_v)-\frac{1}{\dot\tau}b(\tau^{-1}(v),\widetilde{X}^1_v,u^0_v)\right|\mathrm{d}v
\end{aligned}
$$

$$
\begin{aligned}
&+\int_t^s \int_E \left|\beta(\tau^{-1}(v), \widetilde{X}^1_{v-}, u^0_v, e)\right| \left|1-\frac{1}{\dot{\tau}}\right| \Pi(\mathrm{d}e)\mathrm{d}v \\
&+\sup_{r\in[t,s]} \left|\int_t^r \left(\sigma(v, X^0_v, u^0_v) - \frac{1}{\sqrt{\dot{\tau}}}\sigma(\tau^{-1}(v), \widetilde{X}^1_v, u^0_v)\right) \mathrm{d}B_v\right| \\
&+\sup_{r\in[t,s]} \left|\int_t^r \int_E \left(\beta(v, X^0_{v-}, u^0_v, e) - \beta(\tau^{-1}(v), \widetilde{X}^1_{v-}, u^0_v, e)\right) \tilde{\mu}(\mathrm{d}v, \mathrm{d}e)\right|\Bigg)^2 \Bigg|\mathcal{F}_t\Bigg] \\
\leq{}& C\left|X^0_t - \widetilde{X}^1_t\right|^2 + C\mathbb{E}\left[\left(\int_t^s \left(\left|1-\frac{1}{\dot{\tau}}\right| + \left|\tau^{-1}(v) - v\right| + \left|X^0_v - \widetilde{X}^1_v\right|\right)\mathrm{d}v\right)^2 \Bigg|\mathcal{F}_t\right] \\
&+C\mathbb{E}\left[\int_t^s \left(\left|1-\frac{1}{\sqrt{\dot{\tau}}}\right| + \left|\tau^{-1}(v) - v\right| + \left|X^0_v - \widetilde{X}^1_v\right|\right)^2 \mathrm{d}v \Bigg|\mathcal{F}_t\right] \\
\leq{}& C_\delta\left(\left|X^0_t - \widetilde{X}^1_t\right|^2 + |t_0 - t_1|^2\right) + C_\delta \int_t^s \mathbb{E}\left[\left|X^0_v - \widetilde{X}^1_v\right|^2 |\mathcal{F}_t\right]\mathrm{d}v.
\end{aligned}
$$

Finally, from Gronwall's inequality, we have

$$
\mathbb{E}\left[\sup_{s\in[t,T]}\left|X^0_s - \widetilde{X}^1_s\right|^2 |\mathcal{F}_t\right] \leq C_\delta(|t_0 - t_1|^2 + |X^0_t - \widetilde{X}^1_t|^2).
$$

Hence the proof of Lemma 2.5 is complete now. □

After having made comparable X^0 and X^1 by the time change of X^1, we make now Y^0 and Y^1 comparable. For this we put $\widetilde{Y}^1_s = Y^1_{\tau^{-1}(s)}$, $\widetilde{Z}^1_s = \frac{1}{\sqrt{\dot{\tau}}} Z^1_{\tau^{-1}(s)}$ and $\widetilde{U}^1_s = U^1_{\tau^{-1}(s)}$, $s \in [t_0, T]$. Then $(\widetilde{Y}^1, \widetilde{Z}^1, \widetilde{U}^1) = (\widetilde{Y}^1_s, \widetilde{Z}^1_s, \widetilde{U}^1_s)_{s\in[t_0,T]} \in \mathcal{S}^2(t_0, T; \mathbb{R}) \times \mathbb{L}^2(t_0, T; \mathbb{R}^d) \times \mathbb{L}^2(t_0, T; \tilde{\mu}, \mathbb{R})$ is the solution of the BSDE

$$
\begin{aligned}
\widetilde{Y}^1_s ={}& \Phi(\widetilde{X}^1_T) + \int_s^T \frac{1}{\dot{\tau}} f(\tau^{-1}(r), \widetilde{X}^1_r, \widetilde{Y}^1_r, \sqrt{\dot{\tau}}\widetilde{Z}^1_r, \widetilde{U}^1_r, u^0_r)\mathrm{d}r - \int_s^T \widetilde{Z}^1_r \mathrm{d}B_r \\
&- \int_s^T \int_E \widetilde{U}^1_r \left(\tilde{\mu}(\mathrm{d}r, \mathrm{d}e) + \left(1 - \frac{1}{\dot{\tau}}\right)\Pi(\mathrm{d}e)\mathrm{d}r\right), \quad s \in [t_0, T], \qquad (2.10)
\end{aligned}
$$

with respect to the same filtration $\mathbb{F}$ as (Y^0, Z^0, U^0).

For the above BSDE, we have the following a priori estimates which can be proven by a straightforward standard argument.

Lemma 2.8. *Under hypothesis* (**H2**)*, there exists some constant C_δ, only depending on the bounds of σ, b, β, their Lipschitz constants, as well as on $\Pi(E)$ and δ, such that,*

$$
\begin{aligned}
&\mathbb{E}\left[\sup_{s\in[t,T]} |\widetilde{Y}^1_s|^2 + \int_t^T \left|\widetilde{Z}^1_r\right|^2 \mathrm{d}r + \int_t^T \int_E \left|\widetilde{U}^1_r(e)\right|^2 \Pi(\mathrm{d}e)\mathrm{d}r \Bigg|\mathcal{F}_t\right] \\
&\leq C_\delta < +\infty, \quad t \in [t_0, T]. \qquad (2.11)
\end{aligned}
$$

Now we can state the key lemma for proving the joint Lipschitz continuity of V.

Lemma 2.9. *Under our standard assumptions* (**H1**) *and* (**H2**), *we have*

$$\mathbb{E}\left[\sup_{s\in[t,T]}|Y_s^0-\widetilde{Y}_s^1|^2+\int_t^T\left|Z_r^0-\widetilde{Z}_r^1\right|^2\mathrm{d}r+\int_t^T\int_E\left|U_r^0(e)-\widetilde{U}_r^1(e)\right|^2\Pi(\mathrm{d}e)\mathrm{d}r\middle|\mathcal{F}_t\right]$$
$$\le C_\delta\left(|t_0-t_1|^2+|X_t^0-\widetilde{X}_t^1|^2\right),\quad t\in[t_0,T]. \tag{2.12}$$

Proof. First we notice that, for $s\ge t$,

$$\begin{aligned}Y_s^0-\widetilde{Y}_s^1&=\Phi(X_T^0)-\Phi(\widetilde{X}_T^1)\\&\quad+\int_s^T\left(f(r,X_r^0,Y_r^0,Z_r^0,U_r^0,u_r^\lambda)-\frac{1}{\dot\tau}f(\tau_1^{-1}(r),\widetilde{X}_r^1,\widetilde{Y}_r^1,\sqrt{\dot\tau}\widetilde{Z}_r^1,\widetilde{U}_r^1,u_r^\lambda)\right)\mathrm{d}r\\&\quad-\int_s^T\left(Z_r^0-\widetilde{Z}_r^1\right)\mathrm{d}B_r-\int_s^T\int_E\left(U_r^0(e)-\widetilde{U}_r^1(e)\right)\tilde\mu(\mathrm{d}r,\mathrm{d}e)\\&\quad+\int_s^T\int_E\left(1-\frac{1}{\dot\tau}\right)\widetilde{U}_r^1(e)\mathrm{d}r\,\Pi(\mathrm{d}e).\end{aligned}$$

We apply Itô's formula to $|Y_s^0-\widetilde{Y}_s^1|^2$ and, using the boundedness and the Lipschitz continuity of Φ and f, as well as Lemma 2.6, we deduce that

$$\begin{aligned}&|Y_s^0-\widetilde{Y}_s^1|^2+\int_s^T\left|Z_r^0-\widetilde{Z}_r^1\right|^2\mathrm{d}r+\int_s^T\int_E\left|U_r^0(e)-\widetilde{U}_r^1(e)\right|^2\Pi(\mathrm{d}e)\mathrm{d}r\\&\le\left|\Phi(X_T^0)-\Phi(\widetilde{X}_T^1)\right|^2+C\int_s^T\left|X_r^0-\widetilde{X}_r^1\right|^2\mathrm{d}r+C\int_s^T|Y_r^0-\widetilde{Y}_r^1|^2\mathrm{d}r\\&\quad-2\int_s^T\left(Y_r^0-\widetilde{Y}_r^1\right)\left(Z_r^0-\widetilde{Z}_r^1\right)\mathrm{d}B_r+C|t_0-t_1|^2\\&\quad-\int_s^T\int_E\left(2\left(Y_r^0-\widetilde{Y}_r^1\right)\left(U_r^0(e)-\widetilde{U}_r^1(e)\right)+\left|U_r^0(e)-\widetilde{U}_r^1(e)\right|^2\right)\tilde\mu(\mathrm{d}r,\mathrm{d}e)\\&\quad+C|t_0-t_1|^2\int_s^T\left(\left|\widetilde{Z}_r^1\right|^2+\int_E\left|\widetilde{U}_r^1(e)\right|^2\Pi(\mathrm{d}e)\right)\mathrm{d}r.\end{aligned}$$

By taking the conditional expectation on both sides, using Lemma 2.5, the a priori estimate (2.11) and Gronwall's lemma, we obtain, for $t_0\le t\le s\le T$,

$$\mathbb{E}\left[|Y_s^0-\widetilde{Y}_s^1|^2+\int_s^T\left|Z_r^0-\widetilde{Z}_r^1\right|^2\mathrm{d}r+\int_s^T\int_E\left|U_r^0(e)-\widetilde{U}_r^1(e)\right|^2\Pi(\mathrm{d}e)\mathrm{d}r\middle|\mathcal{F}_t\right]$$
$$\le C_\delta\left(|t_0-t_1|^2+|X_t^0-\widetilde{X}_t^1|^2\right).$$

Then the Burkholder–Davis–Gundy inequality allows to show that

$$\mathbb{E}\left[\sup_{s\in[t,T]}|Y_s^0-\widetilde{Y}_s^1|^2+\int_t^T\left|Z_r^0-\widetilde{Z}_r^1\right|^2\mathrm{d}r+\int_t^T\int_E\left|U_r^0(e)-\widetilde{U}_r^1(e)\right|^2\Pi(\mathrm{d}e)\mathrm{d}r\middle|\mathcal{F}_t\right]$$
$$\le C_\delta\left(|t_0-t_1|^2+|X_t^0-\widetilde{X}_t^1|^2\right),\quad t\in[t_0,T].$$

The proof is now complete. □

Now we are ready to give the proof of Theorem 2.2.

Proof of Theorem 2.2. By taking $t = t_0$ in (2.12), we have

$$\begin{aligned}
|Y^0_{t_0} - Y^1_{t_1}|^2 &= |Y^0_{t_0} - \widetilde{Y}^1_{t_0}|^2 \\
&\le C_\delta \left(|t_0 - t_1|^2 + |X^0_{t_0} - \widetilde{X}^1_{t_0}|^2\right) = C_\delta \left(|t_0 - t_1|^2 + |X^0_{t_0} - X^1_{t_1}|^2\right) \\
&= C_\delta \left(|t_0 - t_1|^2 + |x_0 - x_1|^2\right).
\end{aligned}$$

Therefore, from (2.7) and (2.8), we get that

$$\begin{aligned}
V(t_0, x_0) - V(t_1, x_1) &\ge J(t_0, x_0; u^0) - J(t_1, x_1; u^1) - \varepsilon = Y^0_{t_0} - \widetilde{Y}^1_{t_0} - \varepsilon \\
&\ge -C_\delta(|t_0 - t_1| + |x_0 - x_1|) - \varepsilon,
\end{aligned}$$

for some C_δ only depending on δ but not on $(t_0, x_0), (t_1, x_1) \in [0, T - \delta] \times \mathbb{R}^d$. Thus, from the arbitrariness of ε, we deduce that

$$V(t_0, x_0) - V(t_1, x_1) \ge -C_\delta(|t_0 - t_1| + |x_0 - x_1|).$$

Symmetrical argument yields the converse relation, and consequently, the joint Lipschitz continuity of V over $[0, T - \delta] \times \mathbb{R}^d$. □

3. Semiconcavity

We study in this section the semiconcavity property of the viscosity solution V and to extend for this the method of time change and Kulik's transformation used in the preceding section.

For the semiconcavity property, we need more assumptions on the coefficients.

(**H3**) The function $\Phi(x)$ is semiconcave, and $f(\cdot, \cdot, \cdot, \cdot, \cdot, u)$ is semiconcave in $(t, x, y, z, p) \in [0, T] \times \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d \times L^2(E, \mathcal{B}(E), \Pi; \mathbb{R})$, uniformly with respect to $u \in U$, i.e., there exists a constant $C > 0$, such that, for any $\xi_1 \triangleq (t_1, x_1, y_1, z_1, p_1), \xi_2 \triangleq (t_2, x_2, y_2, z_2, p_2)$ in $[0, T] \times \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d \times L^2(E, \mathcal{B}(E), \Pi; \mathbb{R})$, and $\lambda \in [0, 1], u \in U$,

$$\begin{aligned}
&\lambda f(\xi_1, u) + (1 - \lambda) f(\xi_2, u) - f(\lambda\xi_1 + (1 - \lambda)\xi_2, u) \\
&\quad \le C\lambda(1 - \lambda) \Bigg(|t_1 - t_2|^2 + |x_1 - x_2|^2 + |y_1 - y_2|^2 + |z_1 - z_2|^2 \\
&\qquad + \int_E |p_1(e) - p_2(e)|^2 \Pi(\mathrm{d}e)\Bigg).
\end{aligned}$$

(**H4**) The first-order derivatives $\nabla_{t,x} b$, $\nabla_{t,x}\sigma$ and $\nabla_{t,x}\beta$ of b, σ and β with respect to (t, x) exist and are continuous in (t, x, u) and Lipschitz continuous in (t, x), uniformly with respect to u.

(**H5**) There exist two constants $-1 < C_1 < 0$ and $C_2 > 0$ such that, for all $(t, \xi) := (t, x, y, z) \in [0, T] \times \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d$, $u \in U$, and $p, p' \in L^2(E, \mathcal{B}(E), \Pi; \mathbb{R})$,

$$f(t, \xi, p, u) - f(t, \xi, p', u) \le \int_E (p(e) - p'(e))\gamma_t^{\xi,u;p,p'}(e)\Pi(\mathrm{d}e),$$

where $\{\gamma_t^{\xi,u;p,p'}(e)\}_{t\in[0,T]}$ is a measurable function such that, for every $t \in [0, T]$,

$$C_1(1 \wedge |e|) \le \gamma_t^{\xi,u;p,p'}(e) \le C_2(1 \wedge |e|).$$

Our main result in this section is the following.

Theorem 3.1. *Under the assumptions* (**H1**)–(**H5**)*, for every* $\delta \in (0, T)$*, there exists some constant* $C_\delta > 0$ *such that, for all* $(t_0, x_0), (t_1, x_1) \in [0, T-\delta] \times \mathbb{R}^d$*, and for all* $\lambda \in [0, 1]$*:*

$$\lambda V(t_0, x_0) + (1-\lambda) V(t_1, x_1) - V(t_\lambda, x_\lambda) \le C_\delta \lambda (1-\lambda)(|t_0 - t_1|^2 + |x_0 - x_1|^2),$$

where $t_\lambda = \lambda t_0 + (1-\lambda) t_1$, $x_\lambda = \lambda x_0 + (1-\lambda) x_1$.

Remark 3.2. Again as in the case of the Lipschitz continuity, we cannot hope, in general, that the property of semiconcavity holds over the whole domain $[0, T] \times \mathbb{R}^d$. Indeed, let us consider the example given in the preceding section (Remark 2.3). In particular, we have obtained there that

$$V(s, 0) = \mathbb{E}[\Phi(X_T^{s,0})] = \mathbb{E}[-|B_T - B_s|] = -\sqrt{\frac{2}{\pi}}\sqrt{T-s}, \quad s \in [0, T].$$

However, it is easy to check that this function V is not semiconcave in $[0, T] \times \mathbb{R}^d$, but it has this semiconcavity property on $[0, T-\delta] \times \mathbb{R}^d$, for all $\delta > 0$.

The proof of Theorem 3.1 will be based again on the method of time change. But unlike the proof of the Lipschitz property, we have to work here with two time changes. In order to be more precise, for given $\delta > 0$, $(t_0, x_0), (t_1, x_1) \in [0, T-\delta] \times \mathbb{R}^d$, let us consider the both following linear time changes:

$$\tau_i : [t_i, T] \to [t_\lambda, T], \qquad \tau_i(t) = t_\lambda + \frac{T - t_\lambda}{T - t_i}(t - t_i),$$

with the derivatives $\dot{\tau}_i = \frac{T-t_\lambda}{T-t_i}$, $t \in [t_i, T]$, $i = 0, 1$.

For $t = t_\lambda$, we let $B = \{B_s\}_{s \in [t_\lambda, T]}$ be a Brownian motion starting from zero at $t_\lambda : B_{t_\lambda} = 0$. Then $\{W_s^i = \frac{1}{\sqrt{\dot{\tau}_i}} B_{\tau_i(s)}\}_{s \in [t_i, T]}$ is a Brownian motion on $[t_i, T]$, starting from zero at time t_i, $i = 0, 1$. For $t = t_\lambda$, let $\mu(\mathrm{d}r, \mathrm{d}e)$ be our Poisson random measure on $[t_\lambda, T] \times E$ under probability $\mathbb{P}$. Then $\tau_i(\mu)$, $i = 0, 1$, defined as the Kulik transformation of μ,

$$\tau_i(\mu)([t_i, t_i + s] \times \Delta) \triangleq \mu([t_\lambda, \tau_i(t_i + s)] \times \Delta), \quad 0 \le s \le T - t_i, \ \Delta \in \mathcal{B}E$$

is a new Poisson random measure but under probability $\mathbb{Q}_i$, where

$$\frac{\mathrm{d}\mathbb{Q}_i}{\mathrm{d}\mathbb{P}} = \exp\left\{ -\ln\left(\frac{T - t_\lambda}{T - t_i}\right) \mu([t_i, T] \times E) + (t_i - t_\lambda)\Pi(E) \right\}.$$

We denote the corresponding compensated Poisson random measures under $\mathbb{P}$ and $\mathbb{Q}_i$ by $\tilde{\mu}$ and $\widetilde{\tau_i(\mu)}$, $i = 0, 1$, respectively:

$$\tilde{\mu}(\mathrm{d}s, \mathrm{d}e) = \mu(\mathrm{d}s, \mathrm{d}e) - \mathrm{d}s\,\Pi(\mathrm{d}e), \quad (s, e) \in [t_\lambda, T] \times E,$$

and

$$\widetilde{\tau_i(\mu)}(\mathrm{d}s, \mathrm{d}e) = \tau_i(\mu)(\mathrm{d}s, \mathrm{d}e) - \mathrm{d}s\,\Pi(\mathrm{d}e), \quad (s, e) \in [t_i, T] \times E.$$

Let us now fix an arbitrary $u^\lambda \in \mathcal{U}^{B,\mu}(t_\lambda, T)$ (recall the definition of $\mathcal{U}^{B,\mu}(t_\lambda, T)$). Then, obviously, $u_s^i \triangleq u_{\tau_i(s)}^\lambda$, $s \in [t_i, T]$, is an admissible control in $\mathcal{U}^{W^i, \tau_i(\mu)}(t_i, T)$ with respect to W^i and $\tau_i(\mu)$.

We let $\{X_s^\lambda\}_{s\in[t_\lambda,T]}$ be the unique solution of the SDE,

$$\begin{aligned} X_s^\lambda &= x_\lambda + \int_{t_\lambda}^s b(r, X_r^\lambda, u_r^\lambda)\mathrm{d}r + \int_{t_\lambda}^s \sigma(r, X_r^\lambda, u_r^\lambda)\mathrm{d}B_r \\ &\quad + \int_{t_\lambda}^s \int_E \beta(r, X_{r-}^\lambda, u_r^\lambda, e)\tilde{\mu}(\mathrm{d}r, \mathrm{d}e), \quad s \in [t_\lambda, T]. \end{aligned} \tag{3.1}$$

We also make use of the unique solution $\{X_s^i\}_{s\in[t_i,T]}$ of the following SDE,

$$\begin{aligned} X_s^i &= x_i + \int_{t_i}^s b(r, X_r^i, u_r^i)\mathrm{d}r + \int_{t_i}^s \sigma(r, X_r^i, u_r^i)\mathrm{d}W_r^i \\ &\quad + \int_{t_i}^s \int_E \beta(r, X_{r-}^i, u_r^i, e)\widetilde{\tau_i(\mu)}(\mathrm{d}r, \mathrm{d}e), \quad s \in [t_i, T],\ i = 0, 1. \end{aligned} \tag{3.2}$$

As in the preceding section, we associate the forward Eqs. (3.1) and (3.2) with BSDEs. Let $(Y_s^\lambda, Z_s^\lambda, U_s^\lambda)_{s\in[t_\lambda,T]}$ and $(Y_s^i, Z_s^i, U_s^i)_{s\in[t_i,T]}$, $i = 0, 1$, be the unique solutions of the BSDEs

$$\begin{aligned} Y_s^\lambda &= \Phi(X_T^\lambda) + \int_s^T f(r, X_r^\lambda, Y_r^\lambda, Z_r^\lambda, U_r^\lambda, u_r^\lambda)\mathrm{d}r - \int_s^T Z_r^\lambda \mathrm{d}B_r \\ &\quad - \int_s^T \int_E U_r^\lambda(e)\tilde{\mu}(\mathrm{d}r, \mathrm{d}e), \quad s \in [t_\lambda, T], \end{aligned}$$

and

$$\begin{aligned} Y_s^i &= \Phi(X_T^i) + \int_s^T f(r, X_r^i, Y_r^i, Z_r^i, U_r^i, u_r^i)\mathrm{d}r - \int_s^T Z_r^i \mathrm{d}W_r^i \\ &\quad - \int_s^T \int_E U_r^i(e)\widetilde{\tau_i(\mu)}(\mathrm{d}r, \mathrm{d}e), \quad s \in [t_i, T], \end{aligned}$$

respectively. Then from the adaptedness of the solutions Y^λ and Y^i with respect to the filtrations generated by (B, μ) and $(W^i, \tau_i(\mu))$, respectively, we know that $Y_{t_\lambda}^\lambda$ and $Y_{t_i}^i$ are deterministic and equal to the cost functionals $J(t_\lambda, x_\lambda; u^\lambda)$ and $J(t_i, x_i; u^i)$, respectively.

For the proof of Theorem 3.1 it is crucial to estimate $\lambda Y_{t_0}^0 + (1-\lambda)Y_{t_1}^1 - Y_{t_\lambda}^\lambda$, for $\lambda \in (0, 1)$. Since the processes Y^0, Y^1 and Y^λ are solutions of BSDEs over different time intervals, driven by different Brownian motions and different Poisson random measures, we have to make them comparable with the help of the inverse time change.

In a first step we carry out this inverse time change for the forward equations. For this end we introduce the time-changed processes: $\widetilde{X}_s^i = X^i_{\tau_i^{-1}(s)}$, $s \in [t_\lambda, T]$, $i = 0, 1$. Then we have, for $i = 0, 1$,

$$\begin{aligned} \widetilde{X}_s^i &= x_i + \int_{t_\lambda}^s \frac{1}{\dot{\tau}_i} b(\tau_i^{-1}(r), \widetilde{X}_r^i, u_r^\lambda)\mathrm{d}r + \int_{t_\lambda}^s \frac{1}{\sqrt{\dot{\tau}_i}}\sigma(\tau_i^{-1}(r), \widetilde{X}_r^i, u_r^\lambda)\mathrm{d}B_r \\ &\quad + \int_{t_\lambda}^s \int_E \beta(\tau_i^{-1}(r), \widetilde{X}_{r-}^i, u_r^\lambda, e)\Bigg(\widetilde{\mu}(\mathrm{d}r, \mathrm{d}e) \\ &\quad + \left(1 - \frac{1}{\dot{\tau}_i}\right)\Pi(\mathrm{d}e)\mathrm{d}r\Bigg), \quad s \in [t_\lambda, T]. \end{aligned} \tag{3.3}$$

Comparable with Lemma 2.5 but now with arbitrary power $p \geq 2$, we can show the following.

Lemma 3.3. *Let $p \geq 2$. Then there exists a constant $C_{\delta,p}$ depending only on the bounds of σ, b, β, their Lipschitz constants, $\Pi(E)$, δ as well as p, such that, for all $t \in [t_\lambda, T]$,*

$$\mathbb{E}\left[\sup_{t\leq s\leq T}\left|\widetilde{X}_s^0-\widetilde{X}_s^1\right|^p\middle|\mathcal{F}_t\right]\leq C_{\delta,p}\left(\left|\widetilde{X}_t^0-\widetilde{X}_t^1\right|^p+|t_0-t_1|^p\right). \tag{3.4}$$

Moreover, in addition to Lemma 3.3, which gives a kind of "first order estimate", we also have the following kind of "second order estimate". For this we introduce the process $\widetilde{X}^\lambda = \lambda\widetilde{X}^0+(1-\lambda)\widetilde{X}^1$.

Lemma 3.4. *Let $p \geq 2$. There exists a constant $C_{p,\delta}$ depending only on the bounds of σ, b, β, their Lipschitz constants, $\Pi(E)$, δ and p, such that, for all $t \in [t_\lambda, T]$,*

$$\begin{aligned}&\mathbb{E}\left[\sup_{t\leq s\leq T}\left|\widetilde{X}_s^\lambda-X_s^\lambda\right|^p\middle|\mathcal{F}_t\right]\\&\quad\leq C_{p,\delta}\left|\widetilde{X}_t^\lambda-X_t^\lambda\right|^p+C_{p,\delta}(\lambda(1-\lambda))^p\left(|t_0-t_1|^{2p}+\left|\widetilde{X}_t^0-\widetilde{X}_t^1\right|^{2p}\right).\end{aligned} \tag{3.5}$$

For the proof of the above lemmata the reader is referred to the Appendix.

After having applied the inverse time changes to the forward equations, let us do it now for the BSDEs. Thus, for $i = 0, 1$, we introduce the processes $\widetilde{Y}_s^i \triangleq Y^i_{\tau_i^{-1}(s)}$, $\widetilde{Z}_s^i \triangleq \frac{1}{\sqrt{\dot{\tau}_i}}Z^i_{\tau_i^{-1}(s)}$, and $\widetilde{U}_s^i \triangleq U^i_{\tau_i^{-1}(s)}$, $s \in [t_\lambda, T]$. Obviously, $(Y^\lambda, Z^\lambda, U^\lambda)$ and $(\widetilde{Y}^i, \widetilde{Z}^i, \widetilde{U}^i)$ belong to $\mathcal{S}^2(t_\lambda, T; \mathbb{R}) \times \mathbb{L}^2(t_\lambda, T; \mathbb{R}^d) \times \mathbb{L}^2(t_\lambda, T; \tilde{\mu}, \mathbb{R})$, and

$$\begin{aligned}\widetilde{Y}_s^i &= \Phi(\widetilde{X}_T^i)+\int_s^T\frac{1}{\dot{\tau}_i}f(\tau_i^{-1}(r),\widetilde{X}_r^i,\widetilde{Y}_r^i,\sqrt{\dot{\tau}_i}\widetilde{Z}_r^i,\widetilde{U}_r^i,u_r^\lambda)\mathrm{d}r-\int_s^T\widetilde{Z}_r^i\mathrm{d}B_r\\&\quad-\int_s^T\int_E\widetilde{U}_r^i(e)\left(\widetilde{\mu}(\mathrm{d}r,\mathrm{d}e)+\left(1-\frac{1}{\dot{\tau}_i}\right)\Pi(\mathrm{d}e)\mathrm{d}r\right),\qquad s\in[t_\lambda,T].\end{aligned}$$

With the help of standard BSDE estimates we can show the following.

Lemma 3.5. *For $p \geq 2$, there exists some constant C_p only depending on p and the bounds of the coefficients f, Φ, such that, for all $s \in [t_\lambda, T]$, $i = 0, 1$,*

$$\mathbb{E}\left[\sup_{s\leq r\leq T}|\widetilde{Y}^i|^p+\left(\int_s^T|\widetilde{Z}_r^i|^2\mathrm{d}r\right)^{p/2}+\left(\int_s^T\int_E|\widetilde{U}_r^i(e)|^2\Pi(\mathrm{d}e)\mathrm{d}r\right)^{p/2}\middle|\mathcal{F}_s\right]\leq C_p,$$

and

$$\mathbb{E}\left[\sup_{s\leq r\leq T}|Y^\lambda|^p+\left(\int_s^T|Z_r^\lambda|^2\mathrm{d}r\right)^{p/2}+\left(\int_s^T\int_E|U_r^\lambda(e)|^2\Pi(\mathrm{d}e)\mathrm{d}r\right)^{p/2}\middle|\mathcal{F}_s\right]\leq C_p.$$

As the proof uses simple BSDE estimates which by now are standard (see, for instance, [4]), the proof is omitted.

Recall that we have defined $\widetilde{X}^\lambda = \lambda\widetilde{X}^0 + (1-\lambda)\widetilde{X}^1$. In the same manner, we introduce the processes $\widetilde{Y}^\lambda = \lambda\widetilde{Y}^0+(1-\lambda)\widetilde{Y}^1$, $\widetilde{Z}^\lambda = \lambda\widetilde{Z}^0+(1-\lambda)\widetilde{Z}^1$ and $\widetilde{U}^\lambda = \lambda\widetilde{U}^0+(1-\lambda)\widetilde{U}^1$. Then we get that $(\widetilde{Y}^{\lambda}, \widetilde{Z}^{\lambda}, \widetilde{U}^{\lambda}) \in \mathcal{S}^2(t_\lambda, T; \mathbb{R}) \times \mathbb{L}^2(t_\lambda, T; \mathbb{R}^d) \times \mathbb{L}^2(t_\lambda, T; \tilde{\mu}, \mathbb{R})$ is the unique solution of the following BSDE

$$\begin{aligned}
\widetilde{Y}^{\lambda}_s &= \lambda \Phi\left(\widetilde{X}^0_T\right) + (1-\lambda)\Phi\left(\widetilde{X}^1_T\right) - \int_s^T \widetilde{Z}^{\lambda}_r \mathrm{d}B_r - \int_s^T \int_E \widetilde{U}^{\lambda}_r(e)\tilde{\mu}(\mathrm{d}r, \mathrm{d}e) \\
&\quad + \int_s^T \left[\frac{\lambda}{\dot{\tau}_0} f(\tau_0^{-1}(r), \widetilde{X}^0_r, \widetilde{Y}^0_r, \sqrt{\dot{\tau}_0}\widetilde{Z}^0_r, \widetilde{U}^0_r, u^{\lambda}_r)\right. \\
&\quad \left. + \frac{1-\lambda}{\dot{\tau}_1} f(\tau_1^{-1}(r), \widetilde{X}^1_r, \widetilde{Y}^1_r, \sqrt{\dot{\tau}_1}\widetilde{Z}^1_r, \widetilde{U}^1_r, u^{\lambda}_r)\right] \mathrm{d}r - \int_s^T \int_E \left[\lambda\left(1 - \frac{1}{\dot{\tau}_0}\right)\widetilde{U}^0_r(e)\right. \\
&\quad \left. + (1-\lambda)\left(1 - \frac{1}{\dot{\tau}_1}\right)\widetilde{U}^1_r(e)\right] \Pi(\mathrm{d}e)\mathrm{d}r, \quad s \in [t_\lambda, T].
\end{aligned}$$

In analogy to Lemma 3.3 we have for the associated BSDE the following statement, whose proof is postponed in the Appendix.

Lemma 3.6. *For all $p \geq 2$, there exists a constant C_δ depending only on the bounds of σ, b, β, their Lipschitz constants, $\Pi(E), \delta$ and p, such that, for any $t \in [t_\lambda, T]$,*

$$\begin{aligned}
&\mathbb{E}\left[\sup_{t\leq s\leq T} |\widetilde{Y}^0_s - \widetilde{Y}^1_s|^p + \left(\int_t^T \left|\widetilde{Z}^0_s - \widetilde{Z}^1_s\right|^2 \mathrm{d}s\right)^{p/2}\right. \\
&\quad \left. + \left(\int_t^T \int_E \left|\widetilde{U}^0_s(e) - \widetilde{U}^1_s(e)\right|^2 \Pi(\mathrm{d}e)\mathrm{d}s\right)^{p/2} \middle| \mathcal{F}_t\right] \leq C_\delta\left(|\widetilde{X}^0_t - \widetilde{X}^1_t|^p + |t_0 - t_1|^p\right).
\end{aligned}$$

Our objective is to estimate

$$\lambda Y^0_{t_0} + (1-\lambda)Y^1_{t_1} - Y^{\lambda}_{t_\lambda} = \widetilde{Y}^{\lambda}_{t_\lambda} - Y^{\lambda}_{t_\lambda}.$$

For this end some auxiliary processes shall be introduced. So let us introduce the increasing càdlàg processes

$$A_t := |t_0 - t_1| + \sup_{s\in[t_\lambda, t]} |\widetilde{X}^0_s - \widetilde{X}^1_s|,$$

and

$$B_t := \sup_{s\in[t_\lambda, t]} |\widetilde{X}^{\lambda}_s - X^{\lambda}_s|, \quad t \in [t_\lambda, T].$$

For some suitable C and C_δ which will be specified later, we also introduce the increasing càdlàg process

$$D_t = C B_t + C_\delta \lambda(1-\lambda)A_t^2, \quad t \in [t_\lambda, T].$$

We can obtain easily from the Lemmata 3.3 and 3.4 the following estimate for D_t.

Corollary 3.7. *For any $p \geq 2$, there exists a constant C_p such that*

$$\mathbb{E}[|D_s|^p|\mathcal{F}_t] \leq C_p|D_t|^p, \quad \textit{for all } s, t \in [t_\lambda, T], \textit{ with } t \leq s.$$

We observe that, in particular,

$$\mathbb{E}[|D_T|^p] \leq C_p(|t_0 - t_1|^p + |x_0 - x_1|^p) < +\infty, \quad p \geq 2.$$

We let $(\widehat{Y}^\lambda, \widehat{Z}^\lambda, \widehat{U}^\lambda) \in \mathcal{S}^2(t_\lambda, T; \mathbb{R}) \times \mathbb{L}^2(t_\lambda, T; \mathbb{R}^d) \times \mathbb{L}^2(t_\lambda, T; \tilde{\mu}, \mathbb{R})$ be the unique solution of the following BSDE,

$$\begin{aligned}\widehat{Y}^\lambda_s &= \Phi\left(X^\lambda_T\right) + D_T - \int_s^T \widehat{Z}^\lambda_r \mathrm{d}B_r - \int_s^T \int_E \widehat{U}^\lambda_r(e)\tilde{\mu}(\mathrm{d}r, \mathrm{d}e) \\ &\quad + \int_s^T \Big[f(r, X^\lambda_r, \widehat{Y}^\lambda_r - D_r, \widehat{Z}^\lambda_r, \widehat{U}^\lambda_r, u^\lambda_r) + C D_r \\ &\quad + C^0_\delta \lambda(1-\lambda)\left(|t_0 - t_1|^2 \left(1 + |\widetilde{Z}^1_r|^2\right) + |\widetilde{Z}^0_r - \widetilde{Z}^1_r|^2\right) \\ &\quad + \int_E \left|\widetilde{U}^0_r(e) - \widetilde{U}^1_r(e)\right|^2 \Pi(\mathrm{d}e)\Big]\mathrm{d}r, \quad s \in [t_\lambda, T].\end{aligned}$$

The process $\widehat{Y}^\lambda$ stems its importance from the fact that it majorizes $\widetilde{Y}^\lambda$ in a suitable manner. More precisely, we have the following.

Lemma 3.8. *$\widetilde{Y}^\lambda_s \le \widehat{Y}^\lambda_s$, $\mathbb{P}$-a.s., for any $s \in [t_\lambda, T]$.*

For a better readability of the paper, this proof is postponed to the Appendix.

In addition to Lemma 3.8, we also have to estimate the difference between $\widetilde{Y}^\lambda$ and Y^λ. For this we introduce the process $\overline{Y}^\lambda_t = \widehat{Y}^\lambda_t - D_t$, $t \in [t_\lambda, T]$, and we identify $(\overline{Y}^\lambda, \widehat{Z}^\lambda, \widehat{U}^\lambda)$ as the unique solution of the BSDE

$$\begin{aligned}\overline{Y}^\lambda_s &= \Phi\left(X^\lambda_T\right) + \int_s^T \Big[f(r, X^\lambda_r, \overline{Y}^\lambda_t, \widehat{Z}^\lambda_r, \widehat{U}^\lambda_r, u^\lambda_r) + C D_r \\ &\quad + C^0_\delta \lambda(1-\lambda)\left(|t_0 - t_1|^2 \left(1 + |\widetilde{Z}^1_r|^2\right) + |\widetilde{Z}^0_r - \widetilde{Z}^1_r|^2\right) \\ &\quad + \int_E \left|\widetilde{U}^0_r(e) - \widetilde{U}^1_r(e)\right|^2 \Pi(\mathrm{d}e)\Big]\mathrm{d}r - \int_s^T \widehat{Z}^\lambda_r \mathrm{d}B_r - \int_s^T \int_E \widehat{U}^\lambda_r(e)\tilde{\mu}(\mathrm{d}r, \mathrm{d}e) \\ &\quad + \int_s^T \mathrm{d}D_r, \quad s \in [t_\lambda, T].\end{aligned}$$

We observe that we have the following statement, whose proof is given in the Appendix.

Lemma 3.9. *For $t \in [t_\lambda, T]$, we have*

$$\begin{aligned}&\mathbb{E}\Big[\sup_{s\in[t,T]} \left|\overline{Y}^\lambda_s - Y^\lambda_s\right|^2 + \int_t^T \left|\widehat{Z}^\lambda_s - Z^\lambda_s\right|^2 \mathrm{d}s \\ &\quad + \int_t^T \int_E \left|\widehat{U}^\lambda_s(e) - U^\lambda_s(e)\right|^2 \mu(\mathrm{d}s, \mathrm{d}e)\Big|\mathcal{F}_t\Big] \le C_\delta D_t^2. \qquad (3.6)\end{aligned}$$

Now we are ready to give the proof of Theorem 3.1.

Proof of Theorem 3.1. We know from the stochastic interpretation of the viscosity solution V as value function (see (1.4)) that, for any $\lambda \in (0, 1)$ and $\varepsilon > 0$, there exists an admissible control process $u^\lambda \in \mathcal{U}^{B,\mu}(t_\lambda, T)$ such that $Y^\lambda_{t_\lambda} \le V(t_\lambda, x_\lambda) + \varepsilon$. On the other hand, using again (1.4), but now for $\mathcal{U}^{W^i,\tau_i(\mu)}$, we obtain: $V(t_i, x_i) \le Y^i_{t_i}$, $i = 0, 1$. From the Lemmata 3.8 and 3.9 we deduce that

$$\begin{aligned}\lambda V(t_0,x_0)+(1-\lambda)V(t_1,x_1) &\le \lambda Y^0_{t_0}+(1-\lambda)Y^1_{t_1}=\lambda\widetilde{Y}^0_{t_\lambda}+(1-\lambda)\widetilde{Y}^1_{t_\lambda}=\widetilde{Y}^\lambda_{t_\lambda}\\ &\le \widehat{Y}^\lambda_{t_\lambda}=\overline{Y}^\lambda_{t_\lambda}+D_{t_\lambda}\le Y^\lambda_{t_\lambda}+C_\delta D_{t_\lambda}\\ &\le V(t_\lambda,x_\lambda)+C_\delta B_{t_\lambda}+C_\delta\lambda(1-\lambda)A^2_{t_\lambda}+\varepsilon\\ &\le V(t_\lambda,x_\lambda)+C_\delta\lambda(1-\lambda)(|t_0-t_1|^2+|x_0-x_1|^2)+\varepsilon.\end{aligned}$$

Here we have used that $D_{t_\lambda}=CB_{t_\lambda}+C_\delta\lambda(1-\lambda)A^2_{t_\lambda}$ and $B_{t_\lambda}=0$. From the arbitrariness of ε, it follows that

$$\lambda V(t_0,x_0)+(1-\lambda)V(t_1,x_1)\le V(t_\lambda,x_\lambda)+C_\delta\lambda(1-\lambda)(|t_0-t_1|^2+|x_0-x_1|^2).$$

Hence, the semiconcavity of V is proved. □

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Appendix

The Appendix is devoted to the proof of Lemmata 3.3–3.9.

First we give the following lemma, which will be used in what follows. It can be checked by a straightforward computation and, hence, its proof is omitted.

Lemma A.1. *There exists some positive constant C_δ only depending on T and δ such that, for $s\in[t_\lambda,T]$,*

$$\begin{aligned}&|\tau_0^{-1}(s)-\tau_1^{-1}(s)|+\left|\frac{1}{\dot\tau_0}-\frac{1}{\dot\tau_1}\right|+\left|\frac{1}{\sqrt{\dot\tau_0}}-\frac{1}{\sqrt{\dot\tau_1}}\right|\le C_\delta|t_0-t_1|,\\ &\lambda\left|1-\frac{1}{\sqrt{\dot\tau_0}}\right|+(1-\lambda)\left|1-\frac{1}{\sqrt{\dot\tau_1}}\right|\le\frac{1}{2\delta}\lambda(1-\lambda)|t_0-t_1|,\\ &\left|\lambda\left(1-\frac{1}{\sqrt{\dot\tau_0}}\right)+(1-\lambda)\left(1-\frac{1}{\sqrt{\dot\tau_1}}\right)\right|\le\frac{1}{\delta^2}\lambda(1-\lambda)|t_0-t_1|^2.\end{aligned}$$

Moreover, for all $s\in[t_\lambda,T]$,

$$\begin{aligned}&\lambda\left(1-\frac{1}{\dot\tau_0}\right)=-(1-\lambda)\left(1-\frac{1}{\dot\tau_1}\right)=\frac{\lambda(1-\lambda)}{T-t_\lambda}(t_0-t_1),\\ &\lambda\tau_0^{-1}(s)+(1-\lambda)\tau_1^{-1}(s)=s.\end{aligned}$$

We begin with the proof of Lemma 3.3.

Proof of Lemma 3.3. Let us put

$$\Delta\beta(r,e)=\beta(\tau_0^{-1}(r),\widetilde{X}^0_{r-},u^\lambda_r,e)-\beta(\tau_1^{-1}(r),\widetilde{X}^1_{r-},u^\lambda_r,e),$$

for $(r, e) \in [t_\lambda, T] \times E$, and consider, for $t \in [t_\lambda, T]$,

$$M_s = \int_t^s \int_E \Delta\beta(r, e)\tilde{\mu}(\mathrm{d}r, \mathrm{d}e), \quad s \in [t, T].$$

Since $\Delta\beta$ is bounded and predictable, M is a p-integrable martingale, for all $p \geq 2$. We deduce from Itô's formula that,

$$N_s := |M_s|^p - \int_t^s \int_E (|M_r + \Delta\beta(r, e)|^p - |M_r|^p - p|M_r|^{p-2} M_r \Delta\beta(r, e))\Pi(\mathrm{d}e)\mathrm{d}r,\\ s \in [t, T],$$

is a martingale with $N_t = 0$ (see, [5]). Moreover, since β is Lipschitz in (t, x), uniformly with respect to (u, e),

$$\begin{aligned}
&|M_r + \Delta\beta(r, e)|^p - |M_r|^p - p|M_r|^{p-2} M_r \Delta\beta(r, e)\\
&\quad \leq C_p \left(|\Delta\beta(r, e)|^2 |M_r|^{p-2} + |\Delta\beta(r, e)|^p \right)\\
&\quad \leq C_p \left(\left(|\widetilde{X}_r^0 - \widetilde{X}_r^1|^2 + |\tau_0^{-1}(r) - \tau_1^{-1}(r)|^2 \right) |M_r|^{p-2} \right.\\
&\qquad \left. + |\widetilde{X}_r^0 - \widetilde{X}_r^1|^p + |\tau_0^{-1}(r) - \tau_1^{-1}(r)|^p \right)\\
&\quad \leq C_p \left(\left(|\widetilde{X}_r^0 - \widetilde{X}_r^1|^2 + |t_0 - t_1|^2 \right) |M_r|^{p-2} + |\widetilde{X}_r^0 - \widetilde{X}_r^1|^p + |t_0 - t_1|^p \right).
\end{aligned}$$

It follows that, for $s \in [t, T]$,

$$\begin{aligned}
&\mathbb{E}[|M_s|^p | \mathcal{F}_t]\\
&\quad = \mathbb{E}\left[\int_t^s \int_E \left(|M_r + \Delta\beta(r, e)|^p - |M_r|^p - p|M_r|^{p-2} M_r \Delta\beta(r, e) \right) \Pi(\mathrm{d}e)\mathrm{d}r \middle| \mathcal{F}_t \right]\\
&\quad \leq C_p \mathbb{E}\left[\int_t^s \left(\left(|\widetilde{X}_r^0 - \widetilde{X}_r^1|^2 + |t_0 - t_1|^2 \right) |M_r|^{p-2} + |\widetilde{X}_r^0 - \widetilde{X}_r^1|^p + |t_0 - t_1|^p \right) \mathrm{d}r \middle| \mathcal{F}_t \right]\\
&\quad \leq C_p \mathbb{E}\left[\int_t^s \left(|\widetilde{X}_r^0 - \widetilde{X}_r^1|^p + |t_0 - t_1|^p \right) \mathrm{d}r \middle| \mathcal{F}_t \right] + C_p \mathbb{E}\left[\int_t^s |M_r|^p \mathrm{d}r \middle| \mathcal{F}_t \right],
\end{aligned}$$

and from Gronwall's inequality we obtain

$$\mathbb{E}[|M_s|^p | \mathcal{F}_t] \leq C_p \mathbb{E}\left[\int_t^s (|\widetilde{X}_r^0 - \widetilde{X}_r^1|^p + |t_0 - t_1|^p) \mathrm{d}r \middle| \mathcal{F}_t \right], \quad s \in [t, T]. \tag{A.1}$$

Noticing that, for any $t_\lambda \leq t \leq v \leq T$,

$$\begin{aligned}
\widetilde{X}_v^0 - \widetilde{X}_v^1 = {} & \widetilde{X}_t^0 - \widetilde{X}_t^1 + \int_t^v \left(\frac{1}{\dot{\tau}_0} b(\tau_0^{-1}(r), \widetilde{X}_r^0, u_r^\lambda) - \frac{1}{\dot{\tau}_1} b(\tau_1^{-1}(r), \widetilde{X}_r^1, u_r^\lambda) \right) \mathrm{d}r\\
& + \int_t^v \left(\frac{1}{\sqrt{\dot{\tau}_0}} \sigma(\tau_0^{-1}(r), \widetilde{X}_r^0, u_r^\lambda) - \frac{1}{\sqrt{\dot{\tau}_1}} \sigma(\tau_1^{-1}(r), \widetilde{X}_r^1, u_r^\lambda) \right) \mathrm{d}B_r + M_v\\
& + \int_t^v \int_E \left(\left(1 - \frac{1}{\dot{\tau}_0} \right) \beta(\tau_0^{-1}(r), \widetilde{X}_r^0, u_r^\lambda, e) \right.\\
& \left. - \left(1 - \frac{1}{\dot{\tau}_1} \right) \beta(\tau_1^{-1}(r), \widetilde{X}_r^1, u_r^\lambda, e) \right) \Pi(\mathrm{d}e)\mathrm{d}r,
\end{aligned} \tag{A.2}$$

we get, by a standard argument (Recall that $\Pi(E) < \infty$) and Lemma A.1, the existence of a constant $C_{\delta,p}$ such that

$$\begin{aligned}
&\mathbb{E}\left[\sup_{t\le v\le s}|\widetilde{X}^0_v-\widetilde{X}^1_v|^p|\mathcal{F}_t\right]\\
&\le C_{\delta,p}|\widetilde{X}^0_t-\widetilde{X}^1_t|^p+C_p\mathbb{E}\left[\left|\int_t^s\left(\left|\frac{1}{\dot\tau_0}-\frac{1}{\dot\tau_1}\right|+\left|\frac{1}{\dot\tau_1}\right||\widetilde{X}^0_r-\widetilde{X}^1_r|\right)\mathrm{d}r\right|^p\middle|\mathcal{F}_t\right]\\
&\quad+C_{\delta,p}\mathbb{E}\left[\left(\int_t^s\left(\left|\frac{1}{\sqrt{\dot\tau_0}}-\frac{1}{\sqrt{\dot\tau_1}}\right|+\left|\frac{1}{\sqrt{\dot\tau_1}}\right||\widetilde{X}^0_r-\widetilde{X}^1_r|\right)^2\mathrm{d}r\right)^{p/2}\middle|\mathcal{F}_t\right]\\
&\quad+C_{\delta,p}\mathbb{E}\left[|M_s|^p|\mathcal{F}_t\right]+C_{\delta,p}|t_0-t_1|^p.
\end{aligned}$$

Thus, taking into account (A.1), we obtain

$$\begin{aligned}
\mathbb{E}\left[\sup_{t\le v\le s}\left|\widetilde{X}^0_v-\widetilde{X}^1_v\right|^p\middle|\mathcal{F}_t\right]&\le C_{\delta,p}\left(\left|\widetilde{X}^0_t-\widetilde{X}^1_t\right|^p+|t_0-t_1|^p\right)\\
&\quad+C_{\delta,p}\int_t^s\mathbb{E}\left[\sup_{t\le v\le r}\left|\widetilde{X}^0_v-\widetilde{X}^1_v\right|^p\middle|\mathcal{F}_t\right]\mathrm{d}r,\qquad s\in[t_\lambda,T],
\end{aligned}$$

and, finally, Gronwall's lemma yields that

$$\mathbb{E}\left[\sup_{t\le v\le T}\left|\widetilde{X}^0_v-\widetilde{X}^1_v\right|^p\middle|\mathcal{F}_t\right]\le C_{\delta,p}(|\widetilde{X}^0_t-\widetilde{X}^1_t|^p+|t_0-t_1|^p).$$

The proof is complete now. □

Let us now prove Lemma 3.4.

Proof of Lemma 3.4. We observe that, for $s\in[t_\lambda,T]$,

$$\begin{aligned}
&\widetilde{X}^\lambda_s-X^\lambda_s\\
&=\int_{t_\lambda}^s\left(\frac{\lambda}{\dot\tau_0}b(\tau_0^{-1}(r),\widetilde{X}^0_r,u^\lambda_r)+\frac{1-\lambda}{\dot\tau_1}b(\tau_1^{-1}(r),\widetilde{X}^1_r,u^\lambda_r)-b(r,X^\lambda_r,u^\lambda_r)\right)\mathrm{d}r\\
&\quad+\int_{t_\lambda}^s\left(\frac{\lambda}{\sqrt{\dot\tau_0}}\sigma(\tau_0^{-1}(r),\widetilde{X}^0_r,u^\lambda_r)+\frac{1-\lambda}{\sqrt{\dot\tau_1}}\sigma(\tau_1^{-1}(r),\widetilde{X}^1_r,u^\lambda_r)-\sigma(r,X^\lambda_r,u^\lambda_r)\right)\mathrm{d}B_r\\
&\quad+\int_{t_\lambda}^s\int_E\Big(\lambda\beta(\tau_0^{-1}(r),\widetilde{X}^0_{r-},u^\lambda_r,e)+(1-\lambda)\beta(\tau_1^{-1}(r),\widetilde{X}^1_{r-},u^\lambda_r,e)\\
&\quad-\beta(r,X^\lambda_{r-},u^\lambda_r,e)\Big)\tilde\mu(\mathrm{d}r,\mathrm{d}e)+\int_{t_\lambda}^s\int_E\left(\lambda\left(1-\frac{1}{\dot\tau_0}\right)\beta(\tau_0^{-1}(r),\widetilde{X}^0_r,u^\lambda_r,e)\right.\\
&\quad\left.+(1-\lambda)\left(1-\frac{1}{\dot\tau_1}\right)\beta(\tau_1^{-1}(r),\widetilde{X}^1_r,u^\lambda_r,e)\right)\Pi(\mathrm{d}e)\mathrm{d}r.
\end{aligned}\tag{A.3}$$

Taking into account that, as a consequence of the assumptions on the coefficients, we have on one hand that the functions b,σ,β but also $-b,-\sigma,-\beta$ are semiconcave in (t,x), uniformly with respect to u and (u,e), respectively, and that, on the other hand, $\lambda(1-\frac{1}{\dot\tau_0})=-(1-\lambda)(1-\frac{1}{\dot\tau_1})=$

$\frac{\lambda(1-\lambda)}{T-t_\lambda}(t_0-t_1)$ (see Lemma A.1), we deduce

$$\begin{aligned}&\left|\frac{\lambda}{\dot\tau_0}b(\tau_0^{-1}(r),\widetilde X_r^0,u_r^\lambda)+\frac{1-\lambda}{\dot\tau_1}b(\tau_1^{-1}(r),\widetilde X_r^1,u_r^\lambda)-b(r,X_r^\lambda,u_r^\lambda)\right|\\&\quad\le\left|\lambda b(\tau_0^{-1}(r),\widetilde X_r^0,u_r^\lambda)+(1-\lambda)b(\tau_1^{-1}(r),\widetilde X_r^1,u_r^\lambda)-b(r,X_r^\lambda,u_r^\lambda)\right|\\&\qquad+\left|\lambda\left(\frac{1}{\dot\tau_0}-1\right)b(\tau_0^{-1}(r),\widetilde X_r^0,u_r^\lambda)+(1-\lambda)\left(\frac{1}{\dot\tau_1}-1\right)b(\tau_1^{-1}(r),\widetilde X_r^1,u_r^\lambda)\right|\\&\quad\le C_\delta\lambda(1-\lambda)(|t_0-t_1|^2+|\widetilde X_r^0-\widetilde X_r^1|^2)+C_\delta|\widetilde X_r^\lambda-X_r^\lambda|,\quad r\in[t_\lambda,T].\end{aligned}$$

Similarly, we get

$$\begin{aligned}&\left|\frac{\lambda}{\sqrt{\dot\tau_0}}\sigma(\tau_0^{-1}(r),\widetilde X_r^0,u_r^\lambda)+\frac{1-\lambda}{\sqrt{\dot\tau_1}}\sigma(\tau_1^{-1}(r),\widetilde X_r^1,u_r^\lambda)-\sigma(r,X_r^\lambda,u_r^\lambda)\right|\\&\quad\le C_\delta\lambda(1-\lambda)(|t_0-t_1|^2+|\widetilde X_r^0-\widetilde X_r^1|^2)+C_\delta|\widetilde X_r^\lambda-X_r^\lambda|,\quad r\in[t_\lambda,T],\end{aligned}$$

and

$$\begin{aligned}&\left|\lambda\beta(\tau_0^{-1}(r),\widetilde X_r^0,u_r^\lambda,e)+(1-\lambda)\beta(\tau_1^{-1}(r),\widetilde X_r^1,u_r^\lambda,e)-\beta(r,X_r^\lambda,u_r^\lambda,e)\right|\\&\quad\le C_\delta\lambda(1-\lambda)(|t_0-t_1|^2+|\widetilde X_r^0-\widetilde X_r^1|^2)+C_\delta\left|\widetilde X_r^\lambda-X_r^\lambda\right|,\quad r\in[t_\lambda,T].\end{aligned}$$

Moreover, by using again that $\lambda(1-\frac{1}{\dot\tau_0})=-(1-\lambda)(1-\frac{1}{\dot\tau_1})=\frac{\lambda(1-\lambda)}{T-t_\lambda}(t_0-t_1)$, we obtain

$$\begin{aligned}&\left|\lambda\left(1-\frac{1}{\dot\tau_0}\right)\beta(\tau_0^{-1}(r),\widetilde X_r^0,u_r^\lambda,e)+(1-\lambda)\left(1-\frac{1}{\dot\tau_1}\right)\beta(\tau_1^{-1}(r),\widetilde X_r^1,u_r^\lambda,e)\right|\\&\quad\le C_\delta\lambda(1-\lambda)(|t_0-t_1|^2+|\widetilde X_r^0-\widetilde X_r^1|^2),\quad r\in[t_\lambda,T].\end{aligned}$$

Consequently, by combining the above estimates with the argument which has lead to (A.1) in the proof of Lemma 3.3, we get

$$\begin{aligned}&\mathbb{E}\left[\sup_{t\le v\le s}|\widetilde X_v^\lambda-X_v^\lambda|^p\middle|\mathcal{F}_t\right]\\&\quad\le C_{\delta,p}|\widetilde X_t^\lambda-X_t^\lambda|^p+C_\delta(\lambda(1-\lambda))^p\left(|t_0-t_1|^{2p}+\mathbb{E}\left[\int_t^s|\widetilde X_r^0-\widetilde X_r^1|^{2p}\mathrm{d}r\middle|\mathcal{F}_t\right]\right)\\&\qquad+C_{\delta,p}\int_t^s\mathbb{E}\left[\sup_{t\le v\le r}|\widetilde X_v^\lambda-X_v^\lambda|^p\middle|\mathcal{F}_t\right]\mathrm{d}r,\quad t_\lambda\le t\le s\le T,\end{aligned}$$

and, thus, Lemma 3.3 yields

$$\begin{aligned}&\mathbb{E}\left[\sup_{t\le v\le s}|\widetilde X_v^\lambda-X_v^\lambda|^p\middle|\mathcal{F}_t\right]\\&\quad\le C_{\delta,p}|\widetilde X_t^\lambda-X_t^\lambda|^p+C_{\delta,p}(\lambda(1-\lambda))^p(|t_0-t_1|^{2p}+|\widetilde X_t^0-\widetilde X_t^1|^{2p})\\&\qquad+C_{\delta,p}\int_t^s\mathbb{E}\left[\sup_{t\le v\le r}|\widetilde X_v^\lambda-X_v^\lambda|^p\middle|\mathcal{F}_t\right]\mathrm{d}r,\quad t_\lambda\le t\le s\le T.\end{aligned}$$

Hence, Gronwall's inequality gives

$$\begin{aligned}&\mathbb{E}\left[\sup_{t\le s\le T}|\widetilde{X}_s^\lambda - X_s^\lambda|^p\middle|\mathcal{F}_t\right]\\&\quad\le C_{\delta,p}|\widetilde{X}_t^\lambda - X_t^\lambda|^p + C_{\delta,p}(\lambda(1-\lambda))^p(|t_0-t_1|^{2p}+|\widetilde{X}_t^0-\widetilde{X}_t^1|^{2p}).\end{aligned}$$

The proof of Lemma 3.4 is complete. □

We prove now the analogue estimates for our BSDEs stated in Lemma 3.6.

Proof of Lemma 3.6. We notice that, for $t_\lambda \le s \le t \le T$,

$$\begin{aligned}\widetilde{Y}_s^0 - \widetilde{Y}_s^1 &= \Phi(\widetilde{X}_T^0) - \Phi(\widetilde{X}_T^1) + \int_s^T \left(\frac{1}{\dot\tau_0} f(\tau_0^{-1}(r), \widetilde{X}_r^0, \widetilde{Y}_r^0, \sqrt{\dot\tau_0}\widetilde{Z}_r^0, \widetilde{U}_r^0, u_r^\lambda)\right.\\&\quad\left.- \frac{1}{\dot\tau_1} f(\tau_1^{-1}(r), \widetilde{X}_r^1, \widetilde{Y}_r^1, \sqrt{\dot\tau_1}\widetilde{Z}_r^1, \widetilde{U}_r^1, u_r^\lambda)\right)\mathrm{d}r - \int_s^T \left(\widetilde{Z}_r^0 - \widetilde{Z}_r^1\right)\mathrm{d}B_r\\&\quad - \int_s^T\int_E \left(\widetilde{U}_r^0(e) - \widetilde{U}_r^1(e)\right)\widetilde{\mu}(\mathrm{d}r,\mathrm{d}e) - \int_s^T\int_E\left(\left(1-\frac{1}{\dot\tau_0}\right)\widetilde{U}_r^0(e)\right.\\&\quad\left. - \left(1-\frac{1}{\dot\tau_1}\right)\widetilde{U}_r^1(e)\right)\Pi(\mathrm{d}e)\mathrm{d}r. \qquad \text{(A.4)}\end{aligned}$$

By applying Itô's formula to $|\widetilde{Y}_s^0 - \widetilde{Y}_s^1|^2$, we get from the boundedness and the Lipschitz continuity of f that, for $t_\lambda \le s \le t \le T$,

$$\begin{aligned}&|\widetilde{Y}_s^0 - \widetilde{Y}_s^1|^2 + \int_s^T |\widetilde{Z}_r^0 - \widetilde{Z}_r^1|^2\mathrm{d}r + \int_s^T\int_E |\widetilde{U}_r^0(e) - \widetilde{U}_r^1(e)|^2\Pi(\mathrm{d}e)\mathrm{d}r\\&\quad\le |\Phi(\widetilde{X}_T^0) - \Phi(\widetilde{X}_T^1)|^2 + C|t_0-t_1|^2 + C\int_s^T |\widetilde{X}_r^0 - \widetilde{X}_r^1|^2\mathrm{d}r + C\int_s^T|\widetilde{Y}_r^0-\widetilde{Y}_r^1|^2\mathrm{d}r\\&\qquad -2\int_s^T(\widetilde{Y}_r^0 - \widetilde{Y}_r^1)(\widetilde{Z}_r^0-\widetilde{Z}_r^1)\mathrm{d}B_r + |t_0-t_1|^2\int_s^T\left(|\widetilde{Z}_r^1|^2 + \int_E(|\widetilde{U}_r^0(e)|^2\right.\\&\qquad\left. + |\widetilde{U}_r^1(e)|^2)\Pi(\mathrm{d}e)\right)\mathrm{d}r - \int_s^T\int_E (2(\widetilde{Y}_r^0 - \widetilde{Y}_r^1)(\widetilde{U}_r^0(e) - \widetilde{U}_r^1(e))\\&\qquad + |\widetilde{U}_r^0(e) - \widetilde{U}_r^1(e)|^2)\tilde{\mu}(\mathrm{d}r,\mathrm{d}e). \qquad \text{(A.5)}\end{aligned}$$

Then taking the conditional expectation and applying Gronwall's inequality and the Burkholder–Davis–Gundy inequality we obtain from Lemmata 3.3 and 3.5 (recall also the arguments given in the proof of Lemma 3.4) that

$$\begin{aligned}&\mathbb{E}\left[\sup_{t\le r\le T}|\widetilde{Y}_r^0 - \widetilde{Y}_r^1|^2 + \int_t^T|\widetilde{Z}_r^0 - \widetilde{Z}_r^1|^2\mathrm{d}r + \int_t^T\int_E|\widetilde{U}_r^0(e) - \widetilde{U}_r^1(e)|^2\Pi(\mathrm{d}e)\mathrm{d}r\middle|\mathcal{F}_t\right]\\&\quad\le C_\delta(|\widetilde{X}_t^0 - \widetilde{X}_t^1|^2 + |t_0-t_1|^2),\quad t\in[t_\lambda, T].\end{aligned}$$

In particular, we have

$$|\widetilde{Y}_t^0 - \widetilde{Y}_t^1|^2 \le C_\delta(|\widetilde{X}_t^0 - \widetilde{X}_t^1|^2 + |t_0-t_1|^2),\quad t\in[t_\lambda, T].$$

Consequently, from Lemma 3.4, we have, for $p \geq 2$,

$$\mathbb{E}\left[\sup_{t\leq r\leq T}|\widetilde{Y}_r^0-\widetilde{Y}_r^1|^p\middle|\mathcal{F}_t\right]\leq C_\delta(|\widetilde{X}_t^0-\widetilde{X}_t^1|^p+|t_0-t_1|^p),\quad t\in[t_\lambda,T]. \tag{A.6}$$

From (A.5), by using that $\tilde{\mu}(\mathrm{d}r\mathrm{d}e)=\mu(\mathrm{d}r\mathrm{d}e)-\Pi(\mathrm{d}e)\mathrm{d}r$, we get also that

$$\begin{aligned}
&\int_s^T|\widetilde{Z}_r^0-\widetilde{Z}_r^1|^2\mathrm{d}r+\int_s^T\int_E|\widetilde{U}_r^0(e)-\widetilde{U}_r^1(e)|^2\mu(\mathrm{d}r,\mathrm{d}e)\leq|\Phi(\widetilde{X}_T^0)-\Phi(\widetilde{X}_T^1)|^2\\
&\quad+C|t_0-t_1|^2+C\int_s^T|\widetilde{X}_r^0-\widetilde{X}_r^1|^2\mathrm{d}r+C\int_s^T|\widetilde{Y}_r^0-\widetilde{Y}_r^1|^2\mathrm{d}r\\
&\quad-2\int_s^T(\widetilde{Y}_r^0-\widetilde{Y}_r^1)(\widetilde{Z}_r^0-\widetilde{Z}_r^1)\mathrm{d}B_r+|t_0-t_1|^2\int_s^T\Big(|\widetilde{Z}_r^1|^2\\
&\quad+\int_E\left(|\widetilde{U}_r^0(e)|^2+|\widetilde{U}_r^1(e)|^2\right)\Pi(\mathrm{d}e)\Big)\mathrm{d}r-\int_s^T\int_E(2(\widetilde{Y}_r^0-\widetilde{Y}_r^1)(\widetilde{U}_r^0(e)\\
&\quad-\widetilde{U}_r^1(e)))\tilde{\mu}(\mathrm{d}r,\mathrm{d}e).
\end{aligned}$$

Hence, by applying Lemma 3.3 and the Burkholder–Davis–Gundy inequality, we obtain from the above inequality that

$$\begin{aligned}
&\mathbb{E}\left[\left(\int_s^T|\widetilde{Z}_r^0-\widetilde{Z}_r^1|^2\mathrm{d}r\right)^{p/2}+\left(\int_s^T\int_E|\widetilde{U}_r^0(e)-\widetilde{U}_r^1(e)|^2\mu(\mathrm{d}r,\mathrm{d}e)\right)^{p/2}\middle|\mathcal{F}_t\right]\\
&\leq C(|\widetilde{X}_t^0-\widetilde{X}_t^1|^p+|t_0-t_1|^p)+C\mathbb{E}\Bigg[\left(\int_s^T|\widetilde{Y}_r^0-\widetilde{Y}_r^1|^2|\widetilde{Z}_r^0-\widetilde{Z}_r^1|^2\mathrm{d}r\right)^{p/4}\\
&\quad+2\left(\int_s^T\int_E|\widetilde{Y}_r^0-\widetilde{Y}_r^1|^2|\widetilde{U}_r^0(e)-\widetilde{U}_r^1(e)|^2\mu(\mathrm{d}r,\mathrm{d}e)\right)^{p/4}\Bigg|\mathcal{F}_t\Bigg]\\
&\leq C(|\widetilde{X}_t^0-\widetilde{X}_t^1|^p+|t_0-t_1|^p)\\
&\quad+\mathbb{E}\Bigg[C\sup_{s\leq r\leq T}|\widetilde{Y}_r^0-\widetilde{Y}_r^1|^p+\frac{1}{2}\left(\int_s^T|\widetilde{Z}_r^0-\widetilde{Z}_r^1|^2\mathrm{d}r\right)^{p/2}\\
&\quad+\frac{1}{2}\left(\int_s^T\int_E|\widetilde{U}_r^0(e)-\widetilde{U}_r^1(e)|^2\mu(\mathrm{d}r,\mathrm{d}e)\right)^{p/2}\Bigg|\mathcal{F}_t\Bigg].
\end{aligned} \tag{A.7}$$

Combining (A.6) and (A.7), we get

$$\begin{aligned}
&\mathbb{E}\Bigg[\sup_{s\leq r\leq T}|\widetilde{Y}_r^0-\widetilde{Y}_r^1|^p+\left(\int_s^T|\widetilde{Z}_r^0-\widetilde{Z}_r^1|^2\mathrm{d}r\right)^{p/2}\\
&\quad+\left(\int_s^T\int_E|\widetilde{U}_r^0(e)-\widetilde{U}_r^1(e)|^2\mu(\mathrm{d}r,\mathrm{d}e)\right)^{p/2}\Bigg|\mathcal{F}_t\Bigg]\leq C(|\widetilde{X}_t^0-\widetilde{X}_t^1|^p+|t_0-t_1|^p).
\end{aligned} \tag{A.8}$$

In order to replace on the left-hand side of (A.8) the term $\mathbb{E}[(\int_s^T\int_E|\widetilde{U}_r^0(e)-\widetilde{U}_r^1(e)|^2\mu(\mathrm{d}r,\mathrm{d}e))^{p/2}|\mathcal{F}_t]$ by $\mathbb{E}[(\int_s^T\int_E|\widetilde{U}_r^0(e)-\widetilde{U}_r^1(e)|^2\Pi(\mathrm{d}e)\mathrm{d}r)^{p/2}|\mathcal{F}_t]$, we make the following estimate

$$\mathbb{E}\left[\left(\int_s^T\int_E|\widetilde{U}_r^0(e)-\widetilde{U}_r^1(e)|^2\Pi(\mathrm{d}e)\mathrm{d}r\right)^{p/2}\middle|\mathcal{F}_t\right]$$
$$\le(\Pi(E)T)^{\frac{p-2}{2}}\mathbb{E}\left[\int_s^T\int_E|\widetilde{U}_r^0(e)-\widetilde{U}_r^1(e)|^p\Pi(\mathrm{d}e)\mathrm{d}r\middle|\mathcal{F}_t\right]$$
$$=(\Pi(E)T)^{\frac{p-2}{2}}\mathbb{E}\left[\int_s^T\int_E|\widetilde{U}_r^0(e)-\widetilde{U}_r^1(e)|^p\mu(\mathrm{d}r,\mathrm{d}e)\middle|\mathcal{F}_t\right].$$

Let us denote by N the Poisson process $N_s=\mu([t_\lambda,s]\times E)$, $s\in[t_\lambda,T]$, with intensity $\Pi(E)$, the associated sequence of jump times by $\tau_i=\inf\{s\ge t_\lambda : N_s=i\}$, $i\ge 1$, and by $p_i:\{\tau_i\le T\}\to E$ s.t. $\mu(\{(\tau_i,p_i)\})=1$ on $\{\tau_i\le T\}$, the associated sequence of marks. We observe that

$$\int_s^T\int_E|\widetilde{U}_r^0(e)-\widetilde{U}_r^1(e)|^p\mu(\mathrm{d}r,\mathrm{d}e)=\sum_{i\ge1}|\widetilde{U}_{\tau_i}^0(p_i)-\widetilde{U}_{\tau_i}^1(p_i)|^p$$
$$\le\left(\sum_{i\ge1}|\widetilde{U}_{\tau_i}^0(p_i)-\widetilde{U}_{\tau_i}^1(p_i)|^2\right)^{p/2}=\left(\int_s^T\int_E|\widetilde{U}_r^0(e)-\widetilde{U}_r^1(e)|^2\mu(\mathrm{d}r,\mathrm{d}e)\right)^{p/2}.$$

Consequently, from the previous estimate we get

$$\mathbb{E}\left[\left(\int_s^T\int_E|\widetilde{U}_r^0(e)-\widetilde{U}_r^1(e)|^2\Pi(\mathrm{d}e)\mathrm{d}r\right)^{p/2}\middle|\mathcal{F}_t\right]$$
$$\le(\Pi(E)T)^{\frac{p-2}{2}}\mathbb{E}\left[\int_s^T\int_E|\widetilde{U}_r^0(e)-\widetilde{U}_r^1(e)|^p\mu(\mathrm{d}r,\mathrm{d}e)\middle|\mathcal{F}_t\right]$$
$$\le(\Pi(E)T)^{\frac{p-2}{2}}\mathbb{E}\left[\left(\int_s^T\int_E|\widetilde{U}_r^0(e)-\widetilde{U}_r^1(e)|^2\mu(\mathrm{d}r,\mathrm{d}e)\right)^{p/2}\middle|\mathcal{F}_t\right].$$

This latter estimate allows to deduce from (A.8) the wished result. □

Let us now give the proof of Lemma 3.8.

Proof of Lemma 3.8. First we notice that, for $r\in[t_\lambda,T]$,

$$\begin{aligned}
I &:= \frac{\lambda}{\dot{\tau}_0}f\left(\tau_0^{-1}(r),\widetilde{X}_r^0,\widetilde{Y}_r^0,\sqrt{\dot{\tau}_0}\widetilde{Z}_r^0,\widetilde{U}_r^0,u_r^\lambda\right)\\
&\quad+\frac{1-\lambda}{\dot{\tau}_1}f\left(\tau_1^{-1}(r),\widetilde{X}_r^1,\widetilde{Y}_r^1,\sqrt{\dot{\tau}_1}\widetilde{Z}_r^1,\widetilde{U}_r^1,u_r^\lambda\right)\\
&= \lambda f\left(\tau_0^{-1}(r),\widetilde{X}_r^0,\widetilde{Y}_r^0,\sqrt{\dot{\tau}_0}\widetilde{Z}_r^0,\widetilde{U}_r^0,u_r^\lambda\right)\\
&\quad+(1-\lambda)f\left(\tau_1^{-1}(r),\widetilde{X}_r^1,\widetilde{Y}_r^1,\sqrt{\dot{\tau}_1}\widetilde{Z}_r^1,\widetilde{U}_r^1,u_r^\lambda\right)\\
&\quad-\lambda(1-\lambda)\frac{t_1-t_0}{T-t_\lambda}\Big(f(\tau_0^{-1}(r),\widetilde{X}_r^0,\widetilde{Y}_r^0,\sqrt{\dot{\tau}_0}\widetilde{Z}_r^0,\widetilde{U}_r^0,u_r^\lambda)\\
&\quad-f(\tau_1^{-1}(r),\widetilde{X}_r^1,\widetilde{Y}_r^1,\sqrt{\dot{\tau}_1}\widetilde{Z}_r^1,\widetilde{U}_r^1,u_r^\lambda)\Big).
\end{aligned}$$

We also observe that, by applying Lemma A.1,

$$\begin{aligned}&\left|\lambda\sqrt{\dot{\tau}_0}\widetilde{Z}_r^0+(1-\lambda)\sqrt{\dot{\tau}_1}\widetilde{Z}_r^1-\widetilde{Z}_r^\lambda\right|=\left|\lambda\left(\sqrt{\dot{\tau}_0}-1\right)\widetilde{Z}_r^0+(1-\lambda)\left(\sqrt{\dot{\tau}_1}-1\right)\widetilde{Z}_r^1\right|\\&\quad\le\lambda\left|1-\sqrt{\dot{\tau}_0}\right||\widetilde{Z}_r^0-\widetilde{Z}_r^1|+\left|\lambda\left(1-\sqrt{\dot{\tau}_0}\right)+(1-\lambda)\left(1-\sqrt{\dot{\tau}_1}\right)\right||\widetilde{Z}_r^1|\\&\quad\le C_\delta\lambda(1-\lambda)(|t_0-t_1||\widetilde{Z}_r^0-\widetilde{Z}_r^1|+|t_0-t_1|^2|\widetilde{Z}_r^1|).\end{aligned}$$

Thus, by using the estimates from Lemma A.1, the semiconcavity as well as the Lipschitz continuity of f, we have

$$\begin{aligned}I&\le f(r,\widetilde{X}_r^\lambda,\widetilde{Y}_r^\lambda,\lambda\sqrt{\dot{\tau}_0}\widetilde{Z}_r^0+(1-\lambda)\sqrt{\dot{\tau}_1}\widetilde{Z}_r^1,\widetilde{U}_r^\lambda,u_r^\lambda)\\&\quad+C_\delta\lambda(1-\lambda)\Bigg(|t_0-t_1|^2(1+|\widetilde{Z}_r^1|^2)\\&\quad+|\widetilde{X}_r^0-\widetilde{X}_r^1|^2+|\widetilde{Y}_r^0-\widetilde{Y}_r^1|^2+|\widetilde{Z}_r^0-\widetilde{Z}_r^1|^2+\int_E|\widetilde{U}_r^0(e)-\widetilde{U}_r^1(e)|^2\Pi(\mathrm{d}e)\Bigg)\\&\le f(r,\widetilde{X}_r^\lambda,\widetilde{Y}_r^\lambda,\widetilde{Z}_r^\lambda,\widetilde{U}_r^\lambda,u_r^\lambda)+C_\delta\lambda(1-\lambda)\Bigg(|t_0-t_1|^2(1+|\widetilde{Z}_r^1|^2)+|\widetilde{X}_r^0-\widetilde{X}_r^1|^2\\&\quad+|\widetilde{Y}_r^0-\widetilde{Y}_r^1|^2+|\widetilde{Z}_r^0-\widetilde{Z}_r^1|^2+\int_E|\widetilde{U}_r^0(e)-\widetilde{U}_r^1(e)|^2\Pi(\mathrm{d}e)\Bigg).\end{aligned}$$

Finally, by taking into account the Lipschitz continuity of f as well as the definition of D_r, we get

$$\begin{aligned}I&\le f(r,\widetilde{X}_r^\lambda,\widetilde{Y}_r^\lambda-D_r,\widetilde{Z}_r^\lambda,\widetilde{U}_r^\lambda,u_r^\lambda)+C_\delta\lambda(1-\lambda)\Bigg(|t_0-t_1|^2(1+|\widetilde{Z}_r^1|^2)+|\widetilde{X}_r^0-\widetilde{X}_r^1|^2\\&\quad+|\widetilde{Y}_r^0-\widetilde{Y}_r^1|^2+|\widetilde{Z}_r^0-\widetilde{Z}_r^1|^2+\int_E|\widetilde{U}_r^0(e)-\widetilde{U}_r^1(e)|^2\Pi(\mathrm{d}e)\Bigg)+CD_r.\end{aligned}$$

Moreover, thanks to the semiconcavity of Φ, we have

$$\lambda\Phi(\widetilde{X}_T^0)+(1-\lambda)\Phi(\widetilde{X}_T^1)\le\Phi(\widetilde{X}_T^\lambda)+CB_T+C_\delta\lambda(1-\lambda)A_T^2.$$

By virtue of assumption (**H5**), we can use the comparison theorem in [11] (Theorem 2.5) in order to conclude that

$$\widetilde{Y}_s^\lambda\le\widehat{Y}_s^\lambda,\quad\text{for } s\in[t_\lambda,T].$$

The proof of Lemma 3.8 is complete. □

Proof of Lemma 3.9. For some constant $\gamma>0$ which will be specified later, we apply Ito's formula to $e^{\gamma s}(\overline{Y}_s^\lambda-Y_s^\lambda)^2$, and we get

$$\begin{aligned}&\mathrm{d}e^{\gamma s}(\overline{Y}_s^\lambda-Y_s^\lambda)^2\\&\quad=-2e^{\gamma s}(\overline{Y}_s^\lambda-Y_s^\lambda)\Bigg[f(s,X_s^\lambda,\overline{Y}_s^\lambda,\widehat{Z}_s^\lambda,\widehat{U}_s^\lambda,u_s^\lambda)-f(s,X_s^\lambda,Y_s^\lambda,Z_s^\lambda,U_s^\lambda,u_s^\lambda)+CD_s\\&\quad+C_\delta^0\lambda(1-\lambda)(|t_0-t_1|^2(1+|\widetilde{Z}_s^1|^2)+|\widetilde{Z}_s^0-\widetilde{Z}_s^1|^2)\\&\quad+\int_E|\widetilde{U}_s^0(e)-\widetilde{U}_s^1(e)|^2\Pi(\mathrm{d}e)\Bigg]\mathrm{d}s\end{aligned}$$

$$
\begin{aligned}
&+2e^{\gamma s}(\overline{Y}^\lambda_s - Y^\lambda_s)(\widehat{Z}^\lambda_s - Z^\lambda_s)\mathrm{d}B_s + 2e^{\gamma s}(\overline{Y}^\lambda_{s-} - Y^\lambda_{s-})\int_E (\widehat{U}^\lambda_s(e) - U^\lambda_s(e))\tilde{\mu}(\mathrm{d}s,\mathrm{d}e)\\
&-2e^{\gamma s}(\overline{Y}^\lambda_{s-} - Y^\lambda_{s-})\mathrm{d}D_s + e^{\gamma s}(\gamma|\overline{Y}^\lambda_s - Y^\lambda_s|^2 + |\widehat{Z}^\lambda_s - Z^\lambda_s|^2)\mathrm{d}s\\
&+e^{\gamma s}\mathrm{d}[\overline{Y}^\lambda - Y^\lambda]^d_s, \quad s\in[t_\lambda, T],
\end{aligned}
$$

where $[\overline{Y}^\lambda - Y^\lambda]^d_s$ denotes the purely discontinuous part of the quadratic variation of $\overline{Y}^\lambda - Y^\lambda$:

$$
[\overline{Y}^\lambda - Y^\lambda]^d_s = \sum_{t_\lambda < r \le s} (\Delta\overline{Y}^\lambda_r - \Delta Y^\lambda_r)^2, \quad s\in[t_\lambda, T].
$$

We notice that, for $s\in[t_\lambda, T]$,

$$
\begin{aligned}
\int_s^T e^{\gamma r}\mathrm{d}[\overline{Y}^\lambda - Y^\lambda]^d_r &= \sum_{s\le r\le T} e^{\gamma r}(\Delta\overline{Y}^\lambda_r - \Delta Y^\lambda_r)^2\\
&= \sum_{s\le r\le T} e^{\gamma r}\left(\int_E (\widehat{U}^\lambda_r(e) - U^\lambda_r(e))\mu(\{r\},\mathrm{d}e) - \Delta D_r\right)^2\\
&\ge \frac{1}{2}\int_s^T\int_E e^{\gamma r}(\widehat{U}^\lambda_r(e) - U^\lambda_r(e))^2\mu(\mathrm{d}r,\mathrm{d}e) - C_\gamma\int_s^T |\Delta D_r|\mathrm{d}D_r\\
&\ge \frac{1}{2}\int_s^T\int_E e^{\gamma r}(\widehat{U}^\lambda_r(e) - U^\lambda_r(e))^2\mu(\mathrm{d}r,\mathrm{d}e) - C_\gamma(D_T - D_s)^2,
\end{aligned}
$$

where $C_\gamma = e^{\gamma T}$. Hence, by integrating from $s\in[t_\lambda, T]$ to T and taking the conditional expectation on both sides, we deduce that

$$
\begin{aligned}
&e^{\gamma s}|\overline{Y}^\lambda_s - Y^\lambda_s|^2 + \mathbb{E}\bigg[\int_s^T e^{\gamma r}(\gamma|\overline{Y}^\lambda_r - Y^\lambda_r|^2 + |\widehat{Z}^\lambda_r - Z^\lambda_r|^2)\mathrm{d}r\\
&\quad + \frac{1}{2}\int_s^T\int_E e^{\gamma r}|\widehat{U}^\lambda_r(e) - U^\lambda_r(e)|^2\mu(\mathrm{d}r,\mathrm{d}e)\bigg|\mathcal{F}_s\bigg]\\
&\le \mathbb{E}\bigg[\int_s^T 2e^{\gamma r}(\overline{Y}^\lambda_r - Y^\lambda_r)\Big\{f(r, X^\lambda_r, \overline{Y}^\lambda_r, \widehat{Z}^\lambda_r, \widehat{U}^\lambda_r, u^\lambda_r) - f(r, X^\lambda_r, Y^\lambda_r, Z^\lambda_r, U^\lambda_r, u^\lambda_r)\\
&\quad + CD_r + C^0_\delta\lambda(1-\lambda)(|t_0 - t_1|^2(1 + |\widetilde{Z}^1_r|^2) + |\widetilde{Z}^0_r - \widetilde{Z}^1_r|^2)\\
&\quad + \int_E |\widetilde{U}^0_r(e) - \widetilde{U}^1_r(e)|^2\Pi(\mathrm{d}e)\Big\}\mathrm{d}r + \int_s^T 2e^{\gamma r}(\overline{Y}^\lambda_{r-} - Y^\lambda_{r-})\mathrm{d}D_r\\
&\quad + C_\gamma(D_T - D_s)^2\bigg|\mathcal{F}_s\bigg].
\end{aligned}
$$

Then, by a standard argument and from the Lipschitz continuity of f, we get

$$
\begin{aligned}
&e^{\gamma s}|\overline{Y}^\lambda_s - Y^\lambda_s|^2 + \mathbb{E}\bigg[\int_s^T e^{\gamma r}(\gamma\left|\overline{Y}^\lambda_r - Y^\lambda_r\right|^2 + |\widehat{Z}^\lambda_r - Z^\lambda_r|^2)\mathrm{d}r\\
&\quad + \frac{1}{2}\int_s^T\int_E e^{\gamma r}|\widehat{U}^\lambda_r(e) - U^\lambda_r(e)|^2\mu(\mathrm{d}r,\mathrm{d}e)\bigg|\mathcal{F}_s\bigg]\\
&\le \mathbb{E}\bigg[C_K\int_s^T e^{\gamma r}|\overline{Y}^\lambda_r - Y^\lambda_r|^2\mathrm{d}r + \frac{1}{2}\int_s^T e^{\gamma r}|\widehat{Z}^\lambda_r - Z^\lambda_r|^2\mathrm{d}r
\end{aligned}
$$

$$+\frac{1}{4}\int_s^T\int_E e^{\gamma r}|\widehat{U}_r^\lambda(e)-U_r^\lambda(e)|^2\Pi(\mathrm{d}e)\mathrm{d}r\bigg|\mathcal{F}_s\bigg]$$
$$+C_{\delta,\gamma}\mathbb{E}\bigg[\sup_{s\le r\le T}|\overline{Y}_r^\lambda-Y_r^\lambda|D_{s,T}+C_\gamma(D_T-D_s)^2\bigg|\mathcal{F}_s\bigg],$$

where

$$D_{s,T}=D_T+\lambda(1-\lambda)\left(|t_0-t_1|^2\left(1+\int_s^T|\widetilde{Z}_r^1|^2\mathrm{d}r\right)+\int_s^T|\widetilde{Z}_r^0-\widetilde{Z}_r^1|^2\mathrm{d}r\right)$$
$$+\int_s^T\int_E|\widetilde{U}_r^0(e)-\widetilde{U}_r^1(e)|^2\Pi(\mathrm{d}e)\mathrm{d}r.$$

Therefore, by choosing γ large enough and applying Lemma 3.3 and Corollary 3.7, we have the following estimate:

$$|\overline{Y}_s^\lambda-Y_s^\lambda|^2+\mathbb{E}\left[\int_s^T|\widehat{Z}_r^\lambda-Z_r^\lambda|^2\mathrm{d}r+\int_s^T\int_E|\widehat{U}_r^\lambda(e)-U_r^\lambda(e)|^2\mu(\mathrm{d}r,\mathrm{d}e)\bigg|\mathcal{F}_s\right]$$
$$\le C_{\delta,\gamma}\mathbb{E}\bigg[\sup_{s\le r\le T}|\overline{Y}_r^\lambda-Y_r^\lambda|D_{s,T}\bigg|\mathcal{F}_s\bigg]+C_\gamma D_s^2,\quad s\in[t_\lambda,T]. \tag{A.9}$$

From Lemma 3.6 and Corollary 3.7, we get that for $p\ge 2$,

$$\mathbb{E}[|D_{t,T}|^p|\mathcal{F}_t]\le C(|\widetilde{X}_t^0-\widetilde{X}_t^1|^{2p}+|t_0-t_1|^{2p})\le CD_t^p. \tag{A.10}$$

Let $1<p<2$ and $q>2$ be two constants such that $\frac{1}{p}+\frac{1}{q}=1$. Then we have, for $\varepsilon>0$,

$$\begin{aligned}\mathbb{E}\bigg[\sup_{s\le r\le T}|\overline{Y}_r^\lambda-Y_r^\lambda|D_{s,T}\bigg|\mathcal{F}_s\bigg]&\le\mathbb{E}\bigg[\sup_{s\le r\le T}|\overline{Y}_r^\lambda-Y_r^\lambda|^p\bigg|\mathcal{F}_s\bigg]^{\frac{1}{p}}\mathbb{E}[|D_{s,T}|^q|\mathcal{F}_s]^{\frac{1}{q}}\\&\le\varepsilon M_{s,t}^{\frac{2}{p}}+\frac{1}{\varepsilon}\mathbb{E}[|D_{s,T}|^q|\mathcal{F}_s]^{\frac{2}{q}}\\&\le\varepsilon M_{s,t}^{\frac{2}{p}}+\frac{1}{\varepsilon}C_{\delta,q}D_s^2,\quad t_\lambda\le t\le s\le T\end{aligned}$$

where

$$M_{s,t}=\mathbb{E}\bigg[\sup_{t\le r\le T}|\overline{Y}_r^\lambda-Y_r^\lambda|^p\bigg|\mathcal{F}_s\bigg],\quad t_\lambda\le t\le s\le T.$$

Thus, Doob's inequality allows to show that, since $1<p<2$,

$$\begin{aligned}\mathbb{E}\left[\sup_{s\in[t,T]}M_{s,t}^{\frac{2}{p}}|\mathcal{F}_t\right]&\le\left(\frac{2}{2-p}\right)^{\frac{2}{p}}\mathbb{E}\left[M_{T,t}^{\frac{2}{p}}\bigg|\mathcal{F}_t\right]\\&\le\left(\frac{2}{2-p}\right)^{\frac{2}{p}}\mathbb{E}\left[\sup_{s\in[t,T]}|\overline{Y}_s^\lambda-Y_s^\lambda|^2\bigg|\mathcal{F}_t\right],\quad t\in[t_\lambda,T].\end{aligned}$$

Therefore, we can deduce from (A.9) that

$$\mathbb{E}\left[\sup_{s\in[t,T]}|\overline{Y}_s^\lambda-Y_s^\lambda|^2\bigg|\mathcal{F}_t\right]\le C_{\delta,\varepsilon}D_t^2+C_{\delta,\gamma}\varepsilon\left(\frac{2}{2-p}\right)^{\frac{2}{p}}\mathbb{E}\left[\sup_{s\in[t,T]}|\overline{Y}_s^\lambda-Y_s^\lambda|^2\bigg|\mathcal{F}_t\right].$$

By choosing ε small enough such that $C_{\delta,\gamma}\varepsilon\left(\frac{2}{2-p}\right)^{\frac{2}{p}} < 1$, we get

$$\mathbb{E}\left[\sup_{s\in[t,T]}|\overline{Y}_s^\lambda - Y_s^\lambda|^2\Big|\mathcal{F}_t\right] \le C_\delta D_t^2.$$

Hence, it follows easily from (A.9) and (A.10) that

$$\mathbb{E}\left[\sup_{s\in[t,T]}\left|\overline{Y}_s^\lambda - Y_s^\lambda\right|^2 + \int_t^T \left|\widehat{Z}_s^\lambda - Z_s^\lambda\right|^2 \mathrm{d}s + \int_t^T\int_E \left|\widehat{U}_s^\lambda(e) - U_s^\lambda(e)\right|^2 \mu(\mathrm{d}s,\mathrm{d}e)\Big|\mathcal{F}_t\right]$$
$$\le C_\delta D_t^2.$$

The proof of Lemma 3.9 is complete now. □

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